Parametrization and interpolation

- Parametric curves
- Cubic bases
- Arlength parametrization
- Kinetic control

Parametric curves

- General form

$$\boldsymbol{Q}(u) = (x(u), y(u), z(u)) \qquad 0 \leq u \leq 1$$

Cubic curves

- Cubic curves are well suited for interpolation

$$
\begin{aligned}
y(u) &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 \\
y'(u) &= a_1 + 2a_2 u + 3a_3 u^2
\end{aligned}
$$

$$
\begin{array}{rl}
y(0) = y_0 & a = y_0 \\
y'(0) = d_0 & a_1 = d_0 \\
y(1) = y_1 & a_2 + a_3 = y_1 - y_0 - d_0 \\
y'(1) = d_1 & 2a_2 + 3a_3 = d_1 - d_0
\end{array}
$$

Cubic bases and control points

- General form

$$
\begin{aligned}
x(u) &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 \\
y(u) &= b_0 + b_1 u + b_2 u^2 + b_3 u^3 \\
z(u) &= c_0 + c_1 u + c_2 u^2 + c_3 u^3 \\
\boldsymbol{Q}(u) &= \boldsymbol{p}_0 + \boldsymbol{p}_1 u + \boldsymbol{p}_2 u^2 + \boldsymbol{p}_3 u^3 \\
\boldsymbol{Q}(u) &= \sum_{i=0}^{3} \boldsymbol{p}_i b_i(u)
\end{aligned}
$$

The bezier basis

$$
\begin{aligned}
b_0 &= (1-u)^3 \\
b_1 &= 3u(1-u)^2 \\
b_2 &= 3u^2(1-u) \\
b_3 &= u^3
\end{aligned}
$$

$$\boldsymbol{Q}(u) = \boldsymbol{U}\boldsymbol{M}_B\boldsymbol{P}_c = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

The Hermite basis

- Easy control using points and tangents

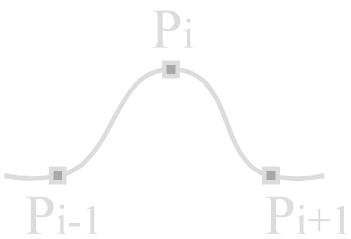
$$\boldsymbol{Q}(u) = \boldsymbol{U}\boldsymbol{M}_B\boldsymbol{P}_c = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{d}_0 \\ \boldsymbol{d}_1 \end{bmatrix}$$

Cardinal splines

- Derived from Hermite: approximate tangents using control points
- Distinguish incoming and outgoing tangent

Catmull-Rom spline

- The tangent at P_i is $P_{i-1}P_{i+1}$



$$\begin{aligned} \boldsymbol{DS}_i = \boldsymbol{DD}_i &= \tfrac{1}{2}(\boldsymbol{P}_{i+1} - \boldsymbol{P}_{i-1}) \\ &\ \tfrac{1}{2}(\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) + \tfrac{1}{2}(\boldsymbol{P}_i - \boldsymbol{P}_{i-1}) \end{aligned}$$

Tension

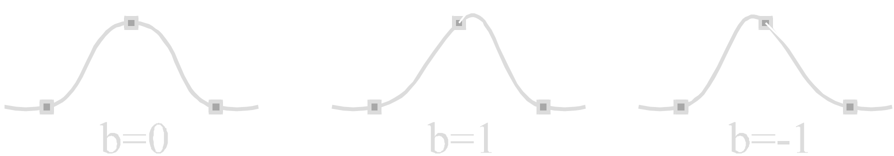
- Tension *t* is responsible for sharpness

$$\boldsymbol{DS}_i = \boldsymbol{DD}_i = \frac{1-t}{2}\left((\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) + (\boldsymbol{P}_i - \boldsymbol{P}_{i-1})\right)$$

Bias

- Bias modifies the slope

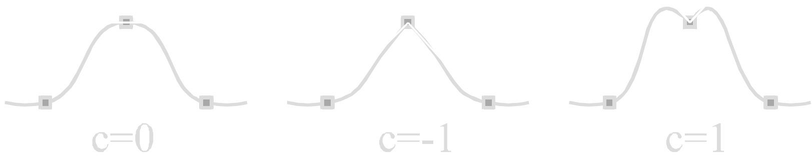
$$\boldsymbol{DS}_i = \boldsymbol{DD}_i = \frac{1-b}{2}(\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) + \frac{1+b}{2}(\boldsymbol{P}_i - \boldsymbol{P}_{i-1})$$



Discontinuity

- Discontinuity splits the tangent in two pieces

$$\begin{aligned} \boldsymbol{DS}_i &= \frac{1+c}{2}(\boldsymbol{P}_{i+1}-\boldsymbol{P}_i)+\frac{1-c}{2}(\boldsymbol{P}_i-\boldsymbol{P}_{i-1}) \\ \boldsymbol{DD}_i &= \frac{1-c}{2}(\boldsymbol{P}_{i+1}-\boldsymbol{P}_i)+\frac{1+c}{2}(\boldsymbol{P}_i-\boldsymbol{P}_{i-1}) \end{aligned}$$



Combination

- Combine tension, discontinuity and bias

$$\begin{aligned} \boldsymbol{DS}_i &= \frac{(1-t)(1+c)(1-b)}{2}(\boldsymbol{P}_{i+1}-\boldsymbol{P}_i)+\frac{(1-t)(1-c)(1+b)}{2}(\boldsymbol{P}_i-\boldsymbol{P}_{i-1}) \\ \boldsymbol{DD}_i &= \frac{(1-t)(1-c)(1-b)}{2}(\boldsymbol{P}_{i+1}-\boldsymbol{P}_i)+\frac{(1-t)(1+c)(1+b)}{2}(\boldsymbol{P}_i-\boldsymbol{P}_{i-1}) \end{aligned}$$

External points

- Use what you prefer !
  - P_0P_1 , $P_{n-1}P_n$
  - combination of P_0P_1 , P_1P_2
  - arbitrary value

Path and velocity

- It is difficult to control Q and dQ/du independently

Arclength parametrization

- We want to control *s(t)*

- Given *s*, we need to compute *u*

$$s = A(u)$$
$$\boldsymbol{Q}(u) = \boldsymbol{Q}(A^{-1}(s))$$

Arclength computation

- To compute *s*, we have to integrate over *u*

$$\begin{aligned} ds &= (dx^2 + dy^2 + dz^2)^{1/2} \\ &= \left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 \right)^{1/2} du \end{aligned}$$

$$s(u) = \int_{u_0}^{u} \left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 \right)^{1/2} du$$

Arclength computation (continued)

- s(u) is not invertible
- use binary search to compute an interval
  u,u´ -> s(u)< s < s(u´)
- interpolate between u and u´

Integration of arclength

- Numerical expression $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$

$$s(u) = \int_{u_0}^{u} (Au^4 + Bu^3 + Cu^2 + Du + E)^{1/2} du$$

$$\begin{aligned} A &= 9(a_x^2 + a_y^2 + a_z^2) \\ B &= 12(a_x b_x + a_y b_y + a_z b_z) \\ C &= 6(a_x c_x + a_y c_y + a_z c_z) + 4(b_x^2 + b_y^2 + b_z^2) \\ D &= 4(b_x c_x + b_y c_y + b_z c_z) \\ E &= c_x^2 + c_y^2 + c_z^2 \end{aligned}$$

Numerical integration

- Use Simpson´s rule

$$\int_a^b f(u)du = \frac{h}{3}(f_0+4f_1+2f_2+4f_3+\cdots+2f_{n-2}+4f_{n-1}+f_n)+O(h^4)$$

$$f_i = f(u_i) \quad u_i = a + \frac{1}{n}(b-a) \quad h = \frac{1}{n}$$

Approximated arclength

- Sum distances between sample points

$$\sum_{j=1}^{i} d_j \quad d_j = \|\boldsymbol{Q}(u_j) - \boldsymbol{Q}(u_{j-1})\|$$

Kinetic control

- Provides control on path and velocity
- The *double interpolant method* uses two curves:
  - Position spline: path of the values
  - Kinetic spline (velocity curve): assign a time to each keyframe. Interpolate inbetween using the specified curve

The double interpolant method

- At given time intervals:
  - compute the arclength *s(t)*
  - find the associated *u* using binary search
  - compute *Q(u)*

Global parametrization

- A global parameter U is mapped to the different intervals.
- Series of three rotations along given axes