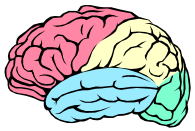
Rendering Equation
Biased MC Methods

Image Synthesis
Torsten Möller

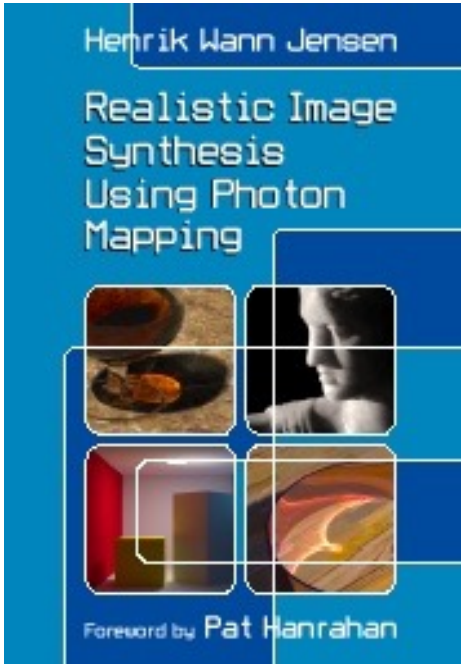
Reading

- Chapter 16.4 of "Physically Based Rendering" by Pharr&Humphreys

- Chapter 19, 20 in "Principles of Digital Image Synthesis," by A. Glassner



- "Realistic Image Synthesis Using Photon Mapping," by H. W. Jensen



Biased vs. Unbiased Algorithm

- unbiased algorithms:
  - Noise very apparent OR we need tons of rays
  - Variance (error) decreases in a predictable way
- Biased approaches $O\left(1/\sqrt{N}\right)$
  - Irradiance caching + photon mapping
  - Smooth results
  - Cheap (only little more computing time than Whitted ray-tracing)

Biased vs. Unbiased Algorithm

- If we find little noise in the image:
  - For unbiased methods
    - can be fairly sure that image is correct
  - For biased methods
    - don't have this assurance
    - Increasing the sampling rate might not get rid of existing artifacts

Irradiance Caching

- Introduced by Greg Ward 1988
- Implemented in RADIANCE
  - Public-domain software
- Exploits smoothness of irradiance
  - Cache and interpolate irradiance estimates

Irradiance Caching

- Indirect lighting changes rather slowly
- Compute indirect light at sparse set of samples
- Interpolate neighboring values from this set of samples

Irradiance Caching

A. How to come up with such a (sparse) set of samples?
B. How to store these samples?
C. How to interpolate from neighbors?

A) Set of Samples

- Find them adaptively
- If there are no good nearby samples then compute a new *irradiance* sample
- Store irradiance (radiance is directionally dependent; i.e. expensive to store):

$$E(p,n) = \int_{H^2(n)} L_i(p,\omega_i)\left|\cos\theta_i\right| d\omega_i$$

- we have: $L_o(p,\omega_o) = \int_{H^2(n)} f_r(p,\omega_o,\omega_i) L_i(p,\omega_i)\left|\cos\theta_i\right| d\omega_i$
- Assuming Lambertian-type surfaces (f=const.)

$$L_o(p,\omega_o) = c \cdot E(p,n)$$

A) Set of Samples

- More glossy surfaces:

$$L_o(p,\omega_o) \approx \left(\int_{H^2(n)} f_r(p,\omega_o,\omega_i)\,d\omega_i\right)\left(\int_{H^2(n)} L_i(p,\omega_i)\left|\cos\theta_i\right|d\omega_i\right)$$

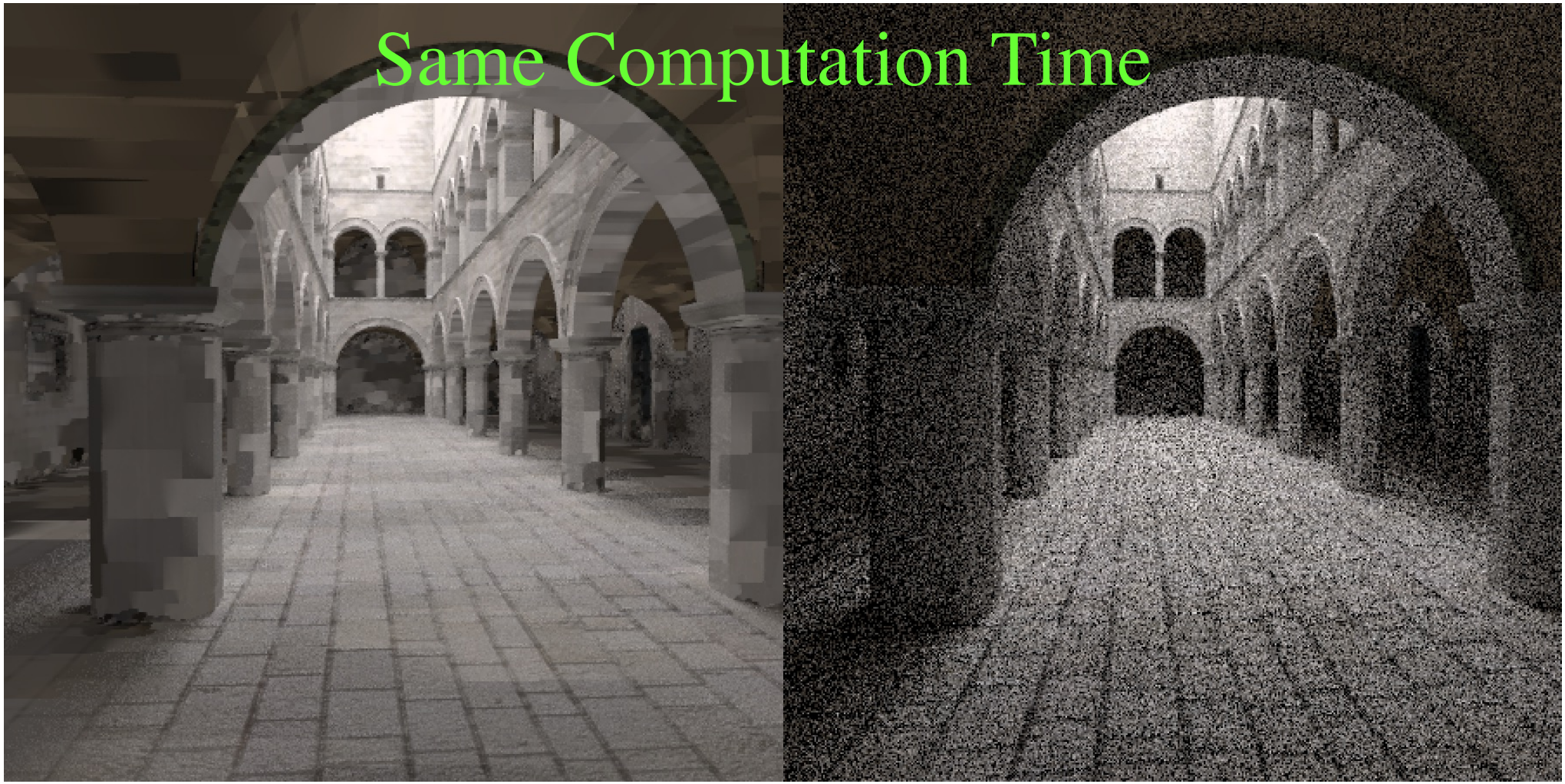
$$= \frac{1}{2}\rho_{hd}(\omega_o)\cdot E(p,n)$$

- $\rho_{hd}(\omega_o)$ hemispherical-directional reflectance

A) Set of Samples

Irradiance chaching

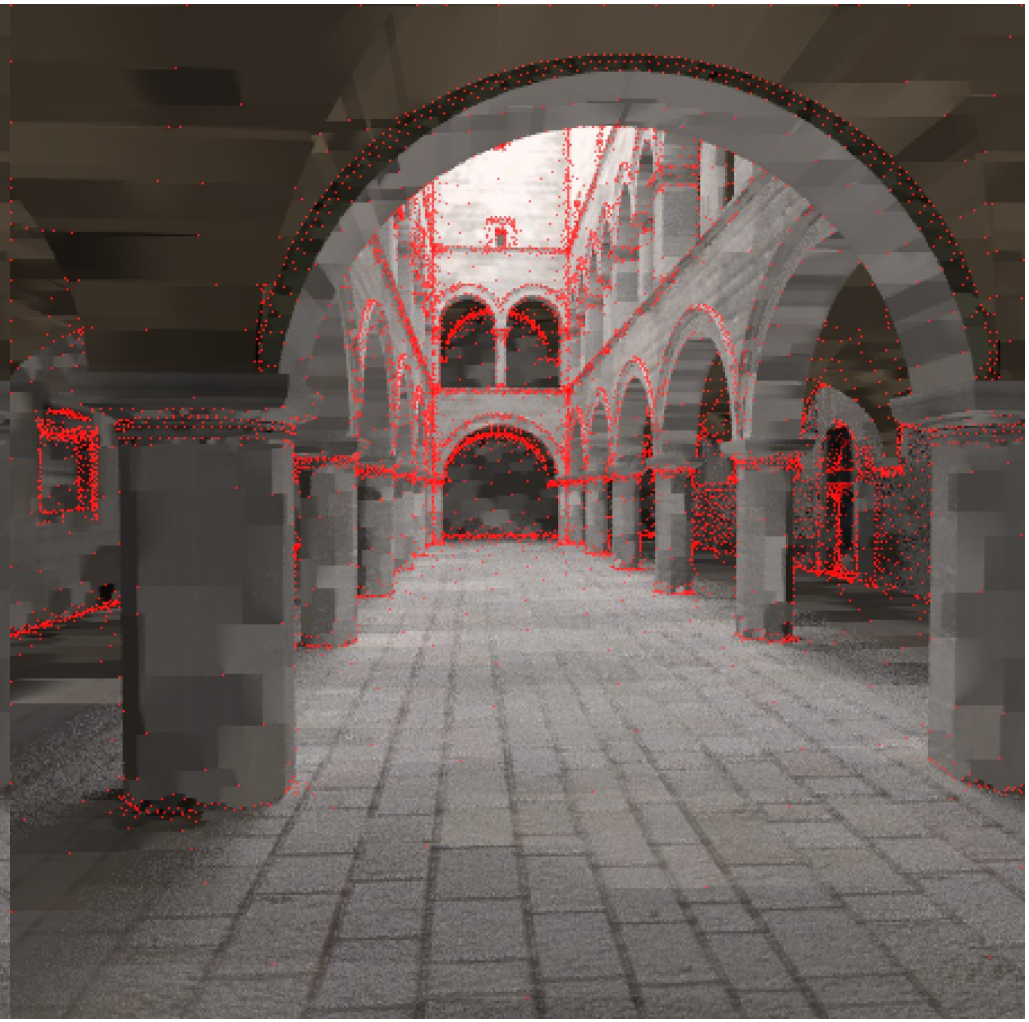
Path Tracing



A) Set of Samples

Irradiance chaching

Irradiance samples used



A) Set of Samples

- High error for specular surface
- Specular BSDF => Whitted integrator
- Diffuse BSDF => irradiance caching
  - Interpolation from known points
  - A path tracing step for new points

$$E(p,n) = \frac{1}{N}\sum_j \frac{L_i(p,\omega_j)\left|\cos\theta_j\right|}{p(\omega_j)}$$

  - Use cos-weighted distribution: $p(\omega) = \frac{\cos\theta}{\pi}$

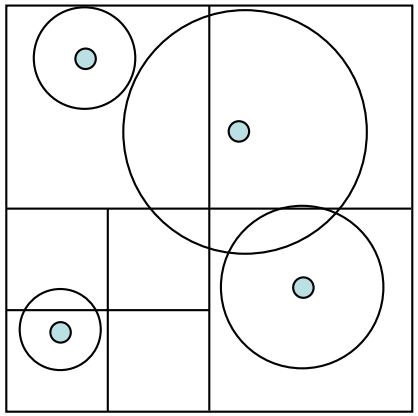
$$E(p,n) = \frac{\pi}{N}\sum_j L_i(p,\omega_j)$$

B) Storing Samples

- Octree data structure
  - Each node stores samples that influence this node (each point has a radius of influence!)
- Radius of influence
  - determined by harmonic mean

$$\frac{N}{\sum_i^N 1/d_i}$$

  - d_i is the distance that the ith ray (used for estimating the irradiance) traveled before intersecting an object
  - Computed during path tracing

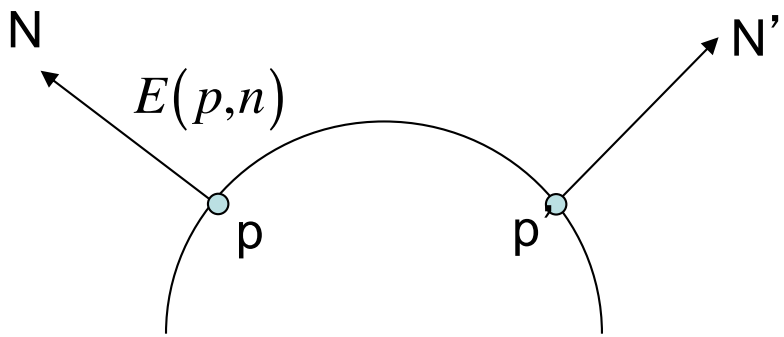


C) Interpolation from neighbors

- Simply weighted sum: $$E = \frac{\sum_i w_i E_i}{\sum_i w_i}$$
- With weights:

$$w_i = \left(\frac{d_{\max} \cdot (N \bullet N') - d}{d_{\max} \cdot (N \bullet N')} \right)^2$$

- Skip samples
  - Normals too different
  - Too far away
  - If in front



Images from “Radiance”

Images from “Radiance”

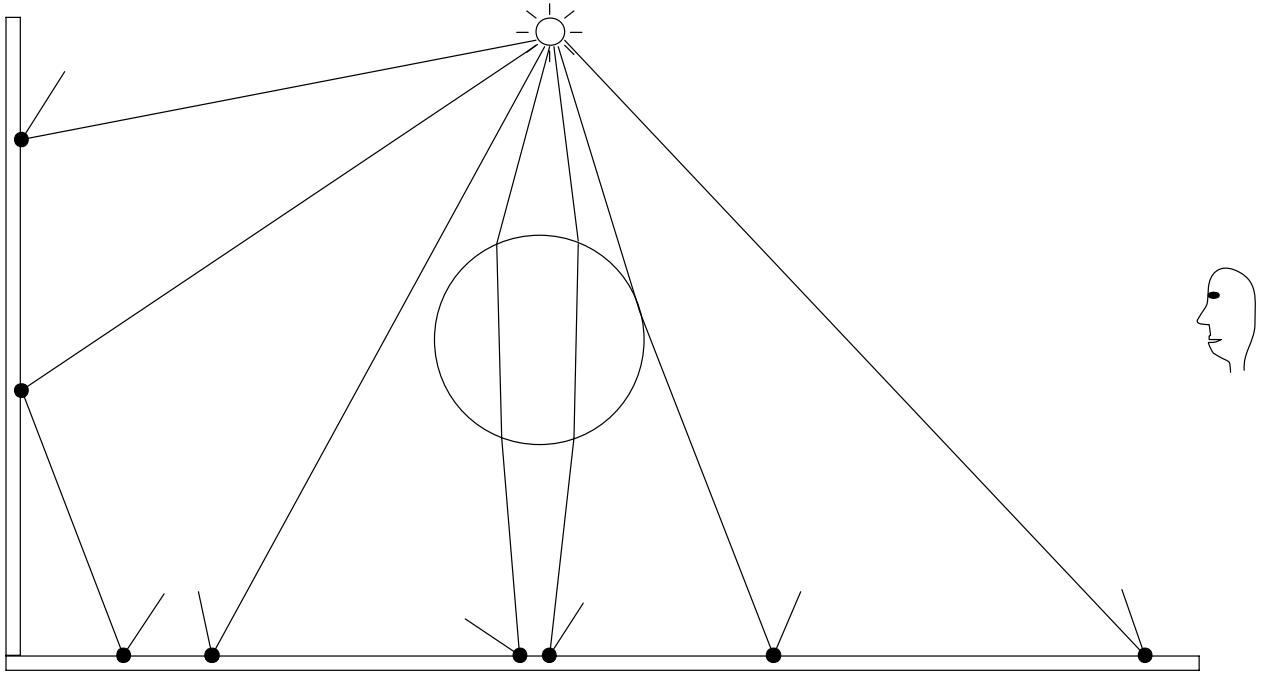
Caching Techniques

- Irradiance caching:
  - Compute irradiance at selected points and interpolate.
- Photon mapping:
  - Trace “photons” from the lights and store them in a photon map, that can be used during rendering.

Photon Mapping

- Basic idea : Density estimation with a discrete density of photons
- 2-step algorithm
  - Photon trace:
    - Simulates the transport of individual photons
    - Photons emitted from source
    - Photons deposited on surfaces
    - Photons reflected from surfaces to other surfaces
  - Rendering
    - Photons collected by rendering

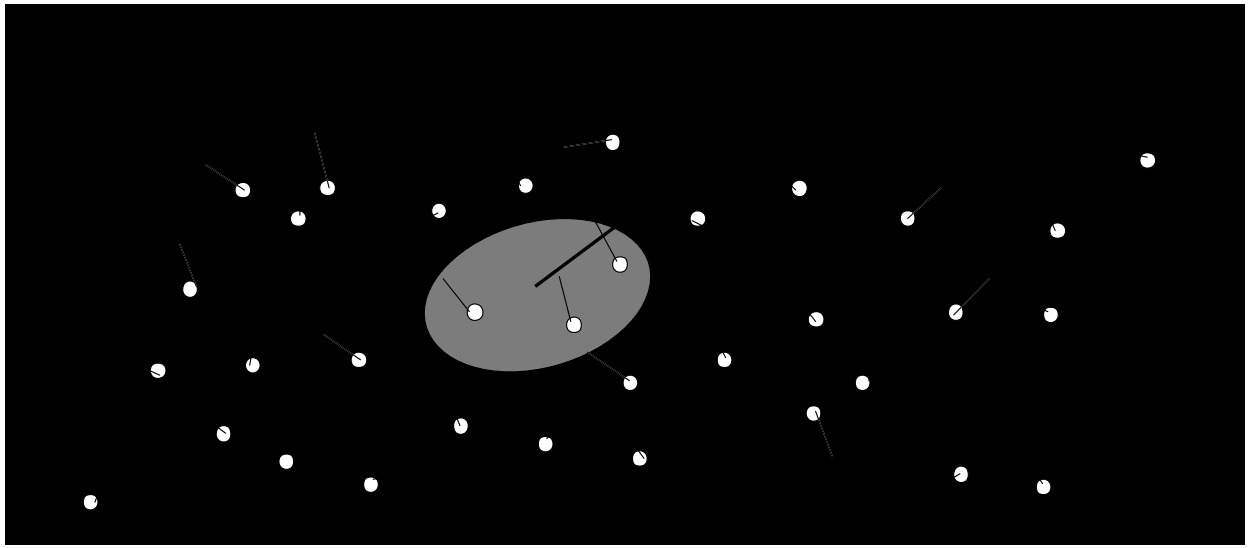
First Pass - Photon Trace



For 100 photons emitted from 100W source,
each photon initially carries 1W.

Propagate this radiant flux through scene using MC methods.

Estimating incident flux



- At any patch of surface, we can estimate the incident flux:

  Just average the contributions of all the photons that hit the patch.

A Photon

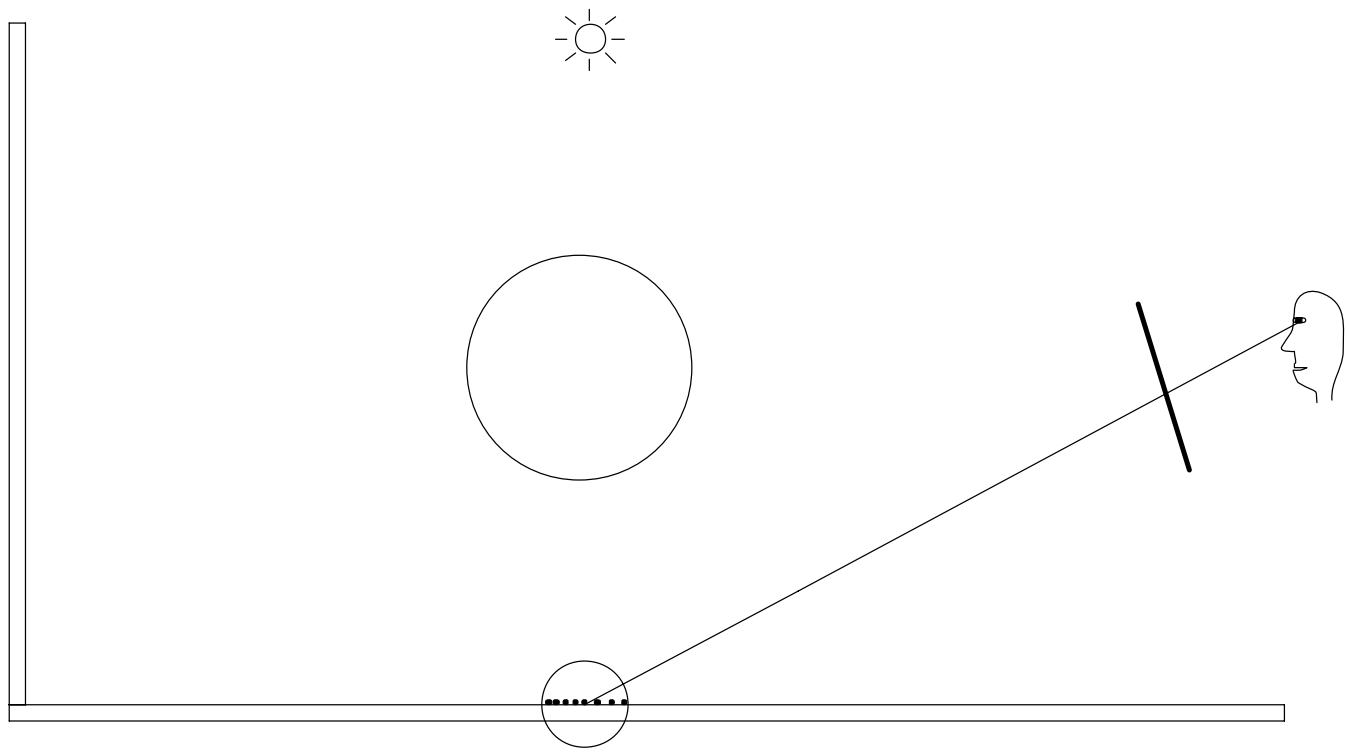
- For each surface interaction, we store:

```
struct photon {
  float x,y,z;            // position
  char power[4];          // power (RGBE)
  char phi, theta;        // incident direction
  short flag;
}
```

Photon Storage

- Store this information about surface interactions in photon map (kd-tree)
- Photon storage is decoupled from geometry

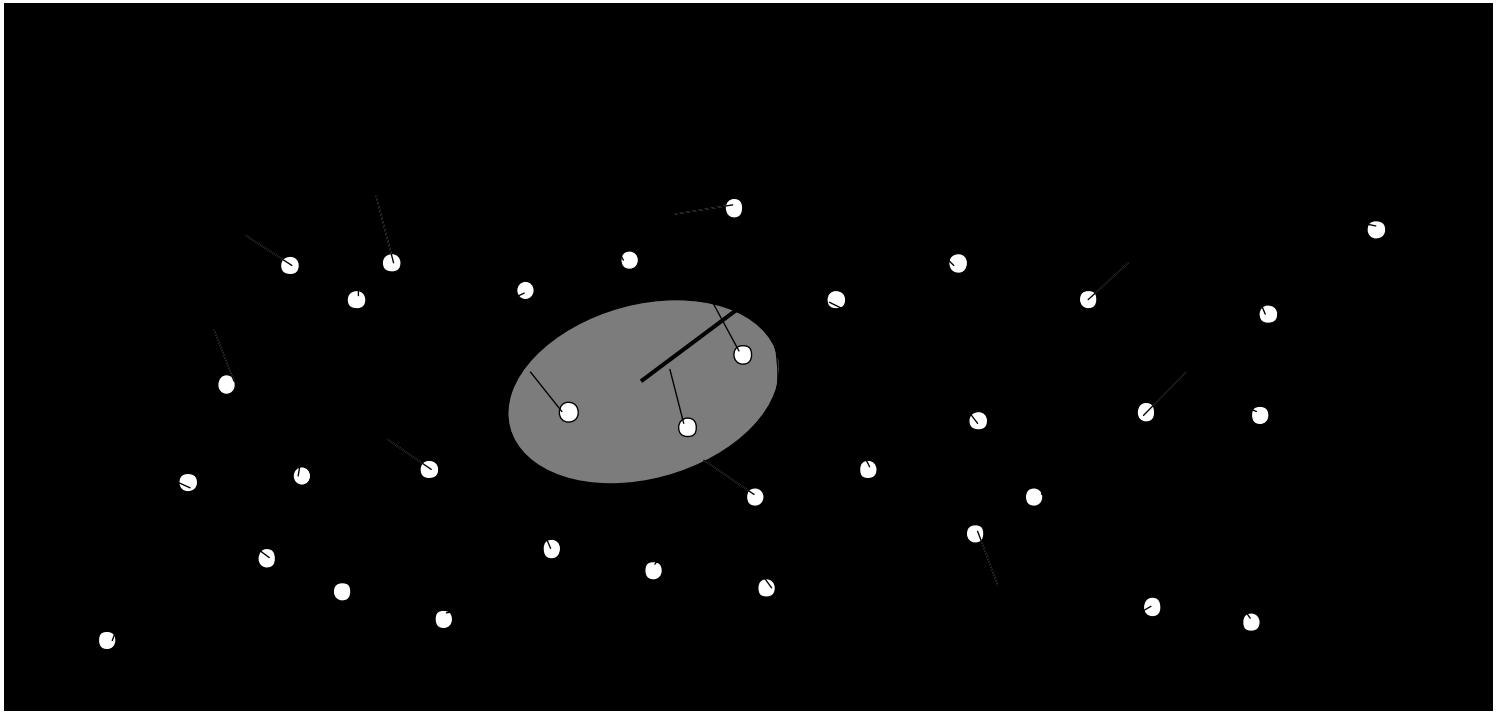
Second Pass - Rendering



- Estimate flux incident at a surface point based on nearby photons.

Radiance Estimate

- Expand ball until it contains some reasonable number of photons.
- Use intersection with plane to estimate area of surface patch.



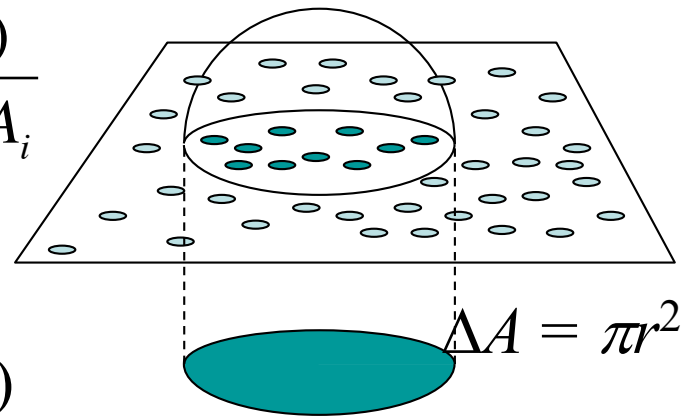
Radiance Estimate

- Recall the reflected radiance equation

$$L_o(p,\omega_o) = \int_{S^2} f_r(p,\omega_i,\omega_o) L_i(p,\omega_i) |\cos\theta_i| d\omega_i$$

- Convert incident radiance into incident flux

$$L_i(p,\omega_i) = \frac{d^2\Phi_i(p,\omega_i)}{|\cos\theta_i| d\omega_i dA_i}$$



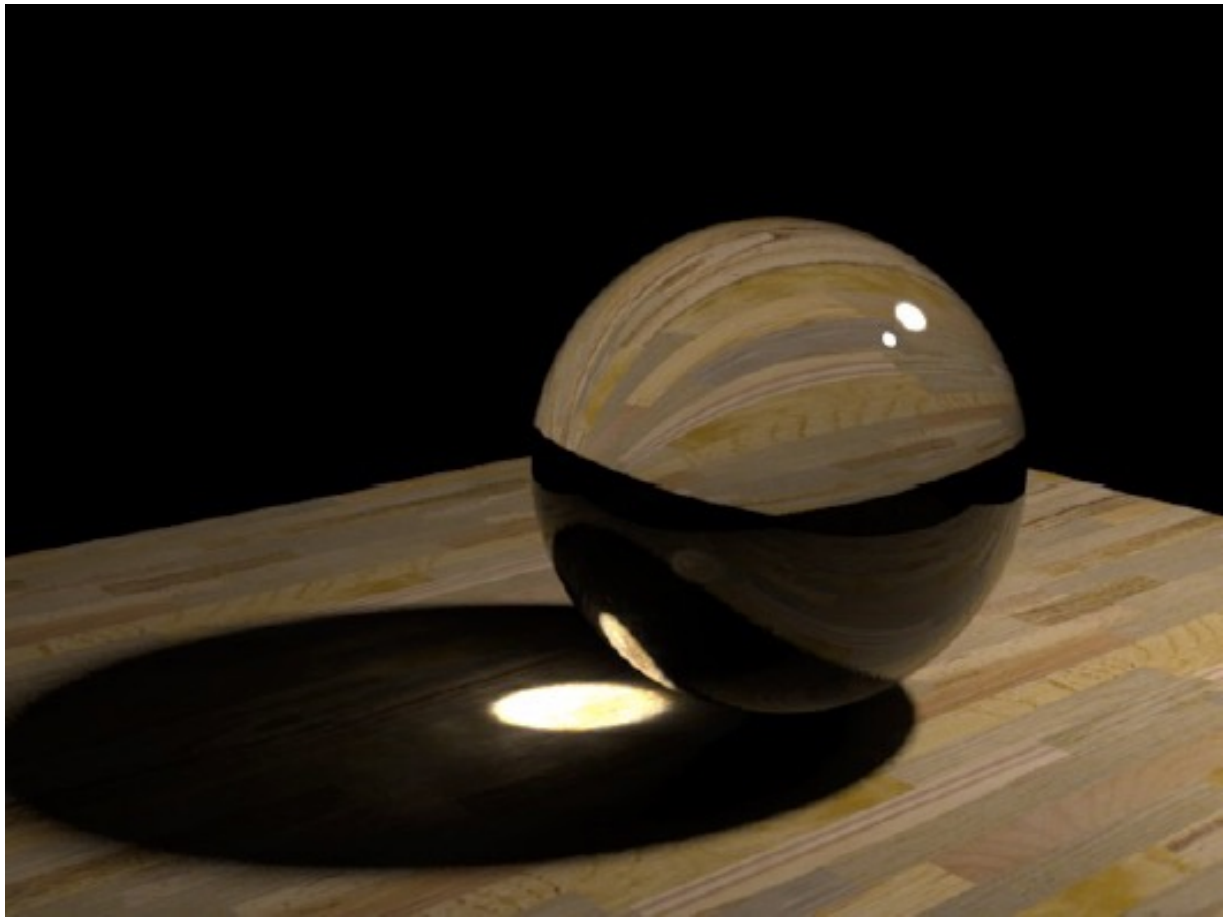
- Reflected radiance in terms of incident flux

$$L_o(p,\omega_o) = \int_{S^2} f_r(p,\omega_i,\omega_o) \frac{d^2\Phi_i(p,\omega_i)}{dA_i}$$

- Numerically $$L_o(p,\omega_o) \approx \frac{1}{\Delta A} \sum_{p=1}^{n} f_r(p,\omega_p,\omega_o) \Delta\Phi_p(p,\omega_p)$$

Caustic from a Glass Sphere

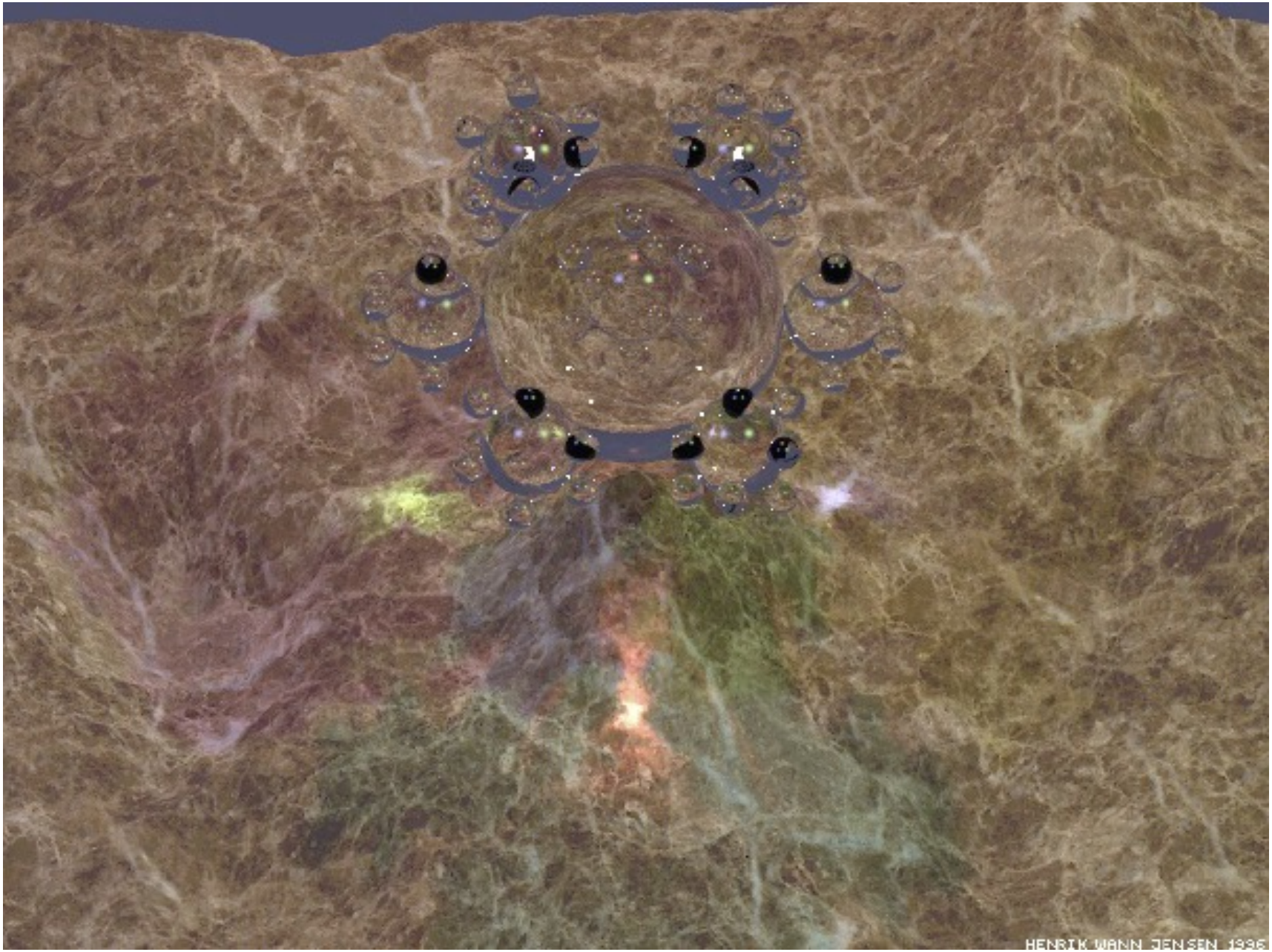
- Photon Mapping: 10000 photons / 50 photons in radiance estimate



Caustic from a Glass Sphere

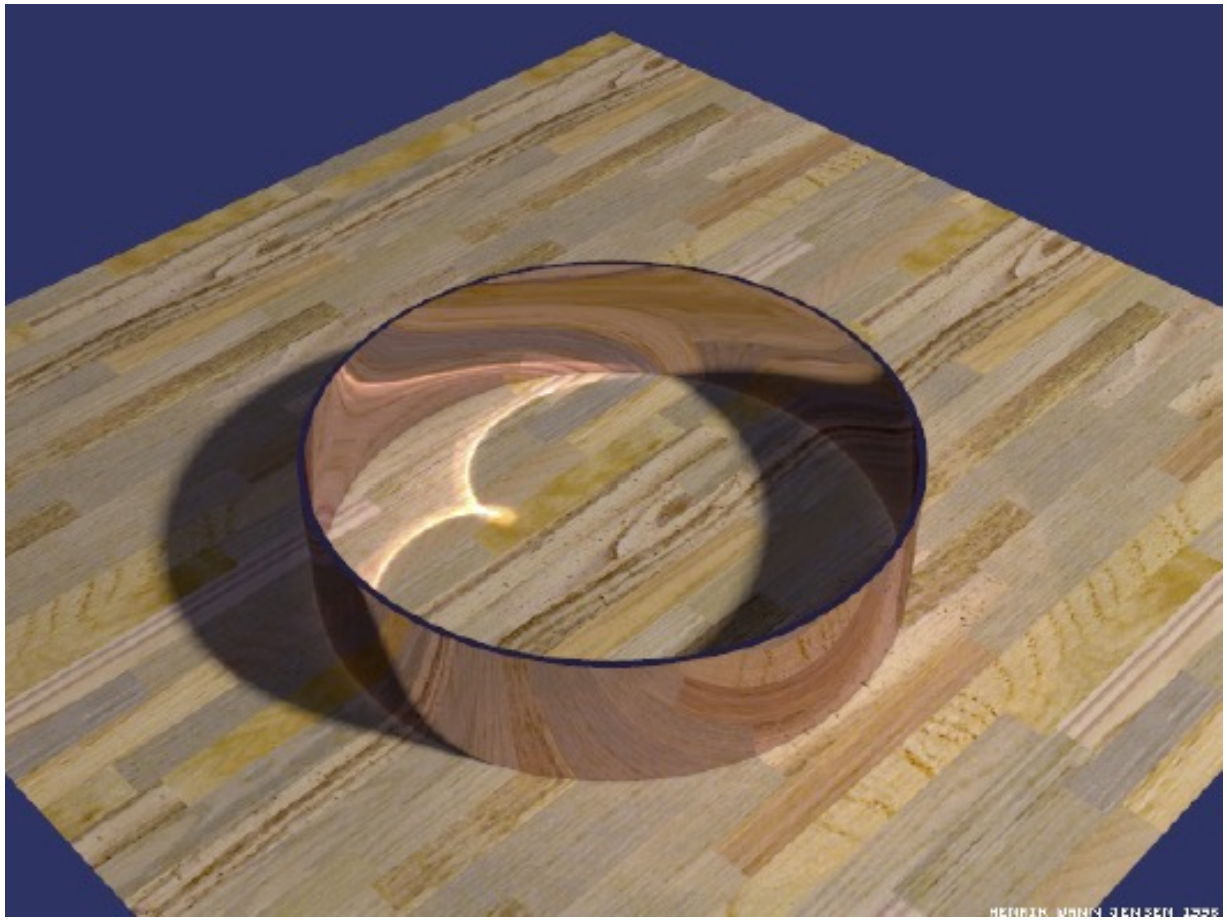
- Path Tracing: 1000 paths/pixel

Sphereflake Caustic



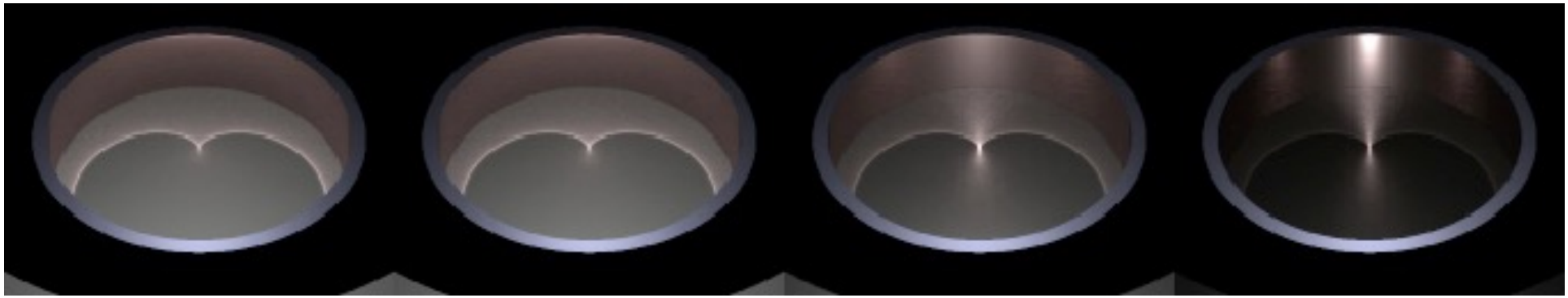
Reflection Inside A Metal Ring

- 50000 photons / 50 photons in radiance estimate



Caustics On Glossy Surfaces

- 340000 photons / 100 photons in radiance estimate

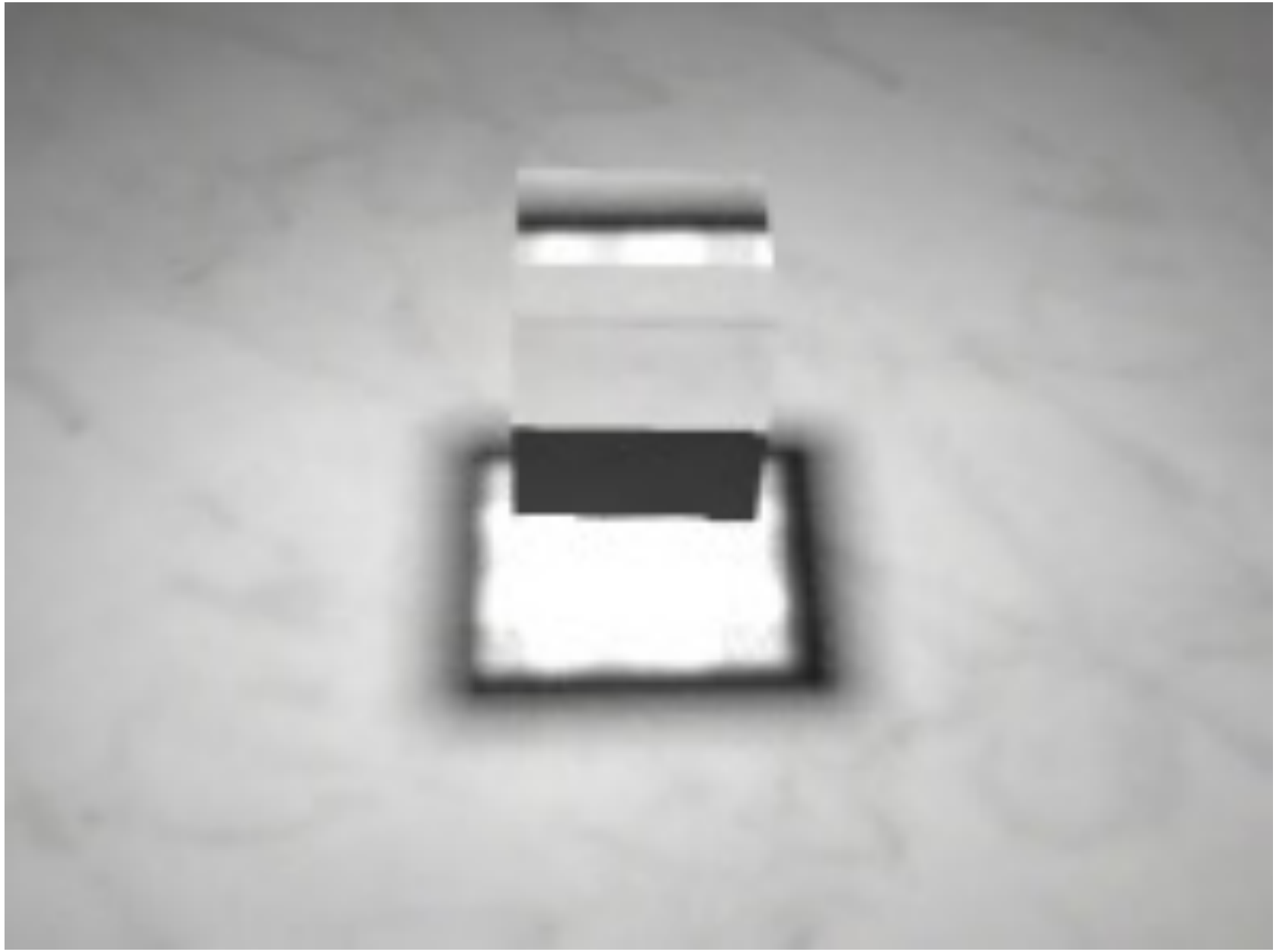


HDR environment illumination

- Using lightprobe from www.debevec.org

Cognac Glass

Cube Caustic

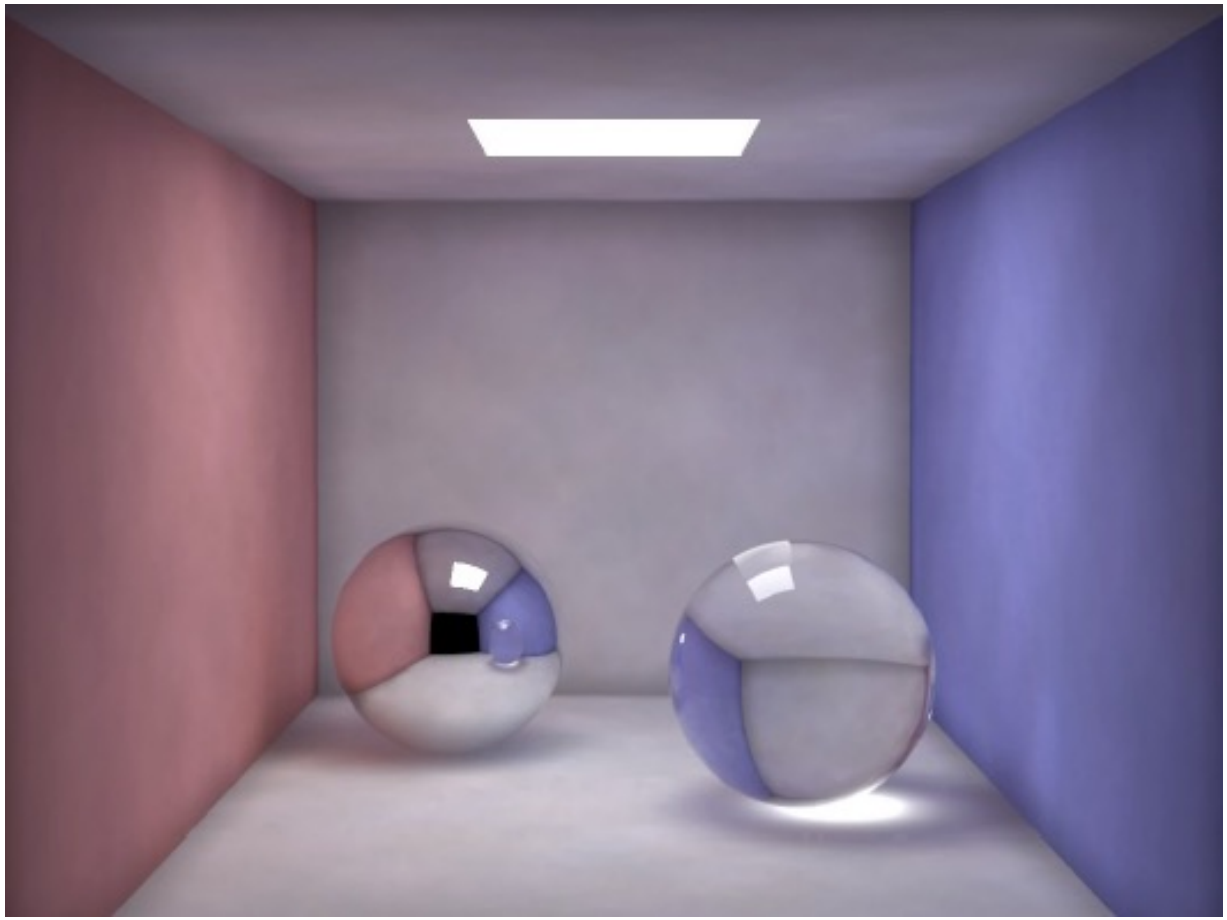


Global Illumination

- 100000 photons / 50 photons in radiance estimate

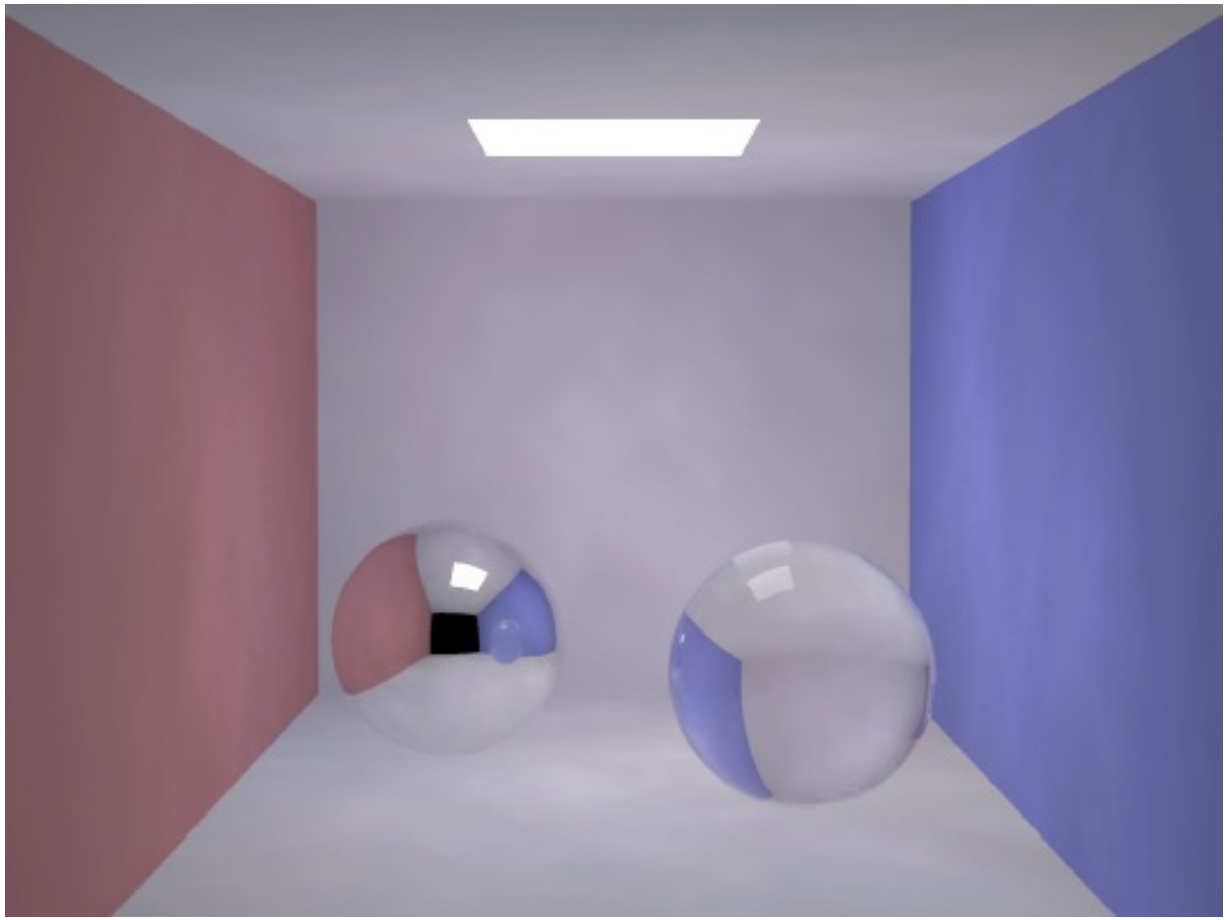
Global Illumination

- 500000 photons / 500 photons in radiance estimate



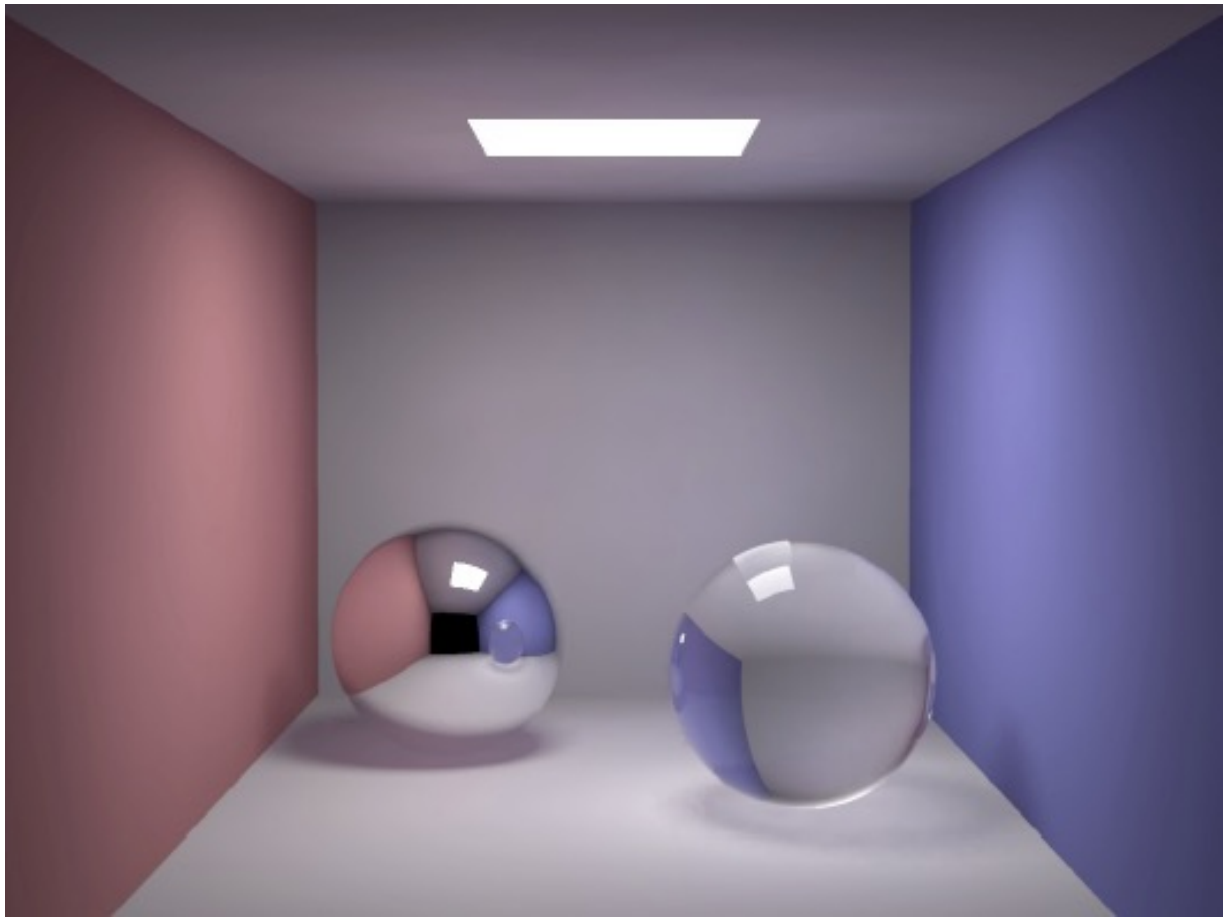
Fast estimate

- 200 photons / 50 photons in radiance estimate



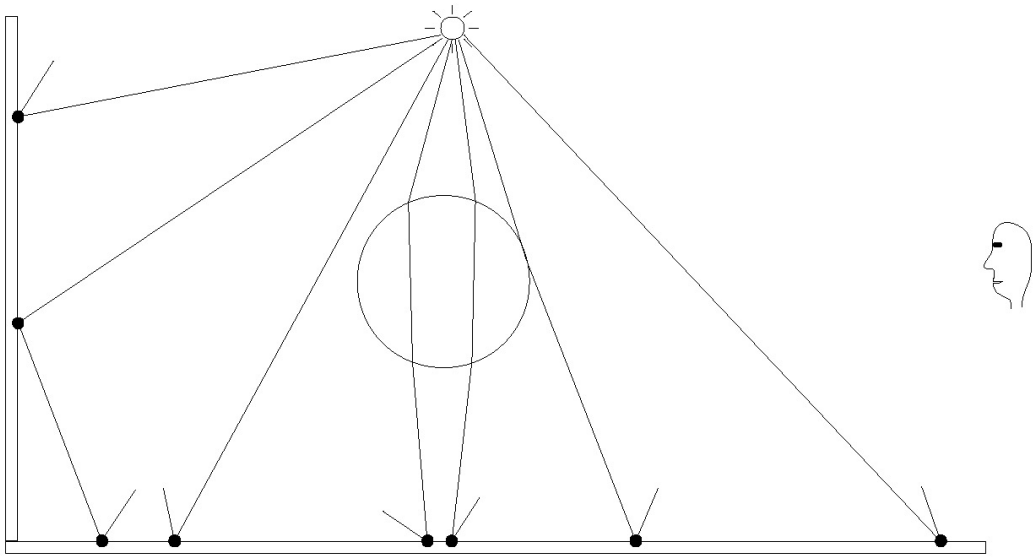
Indirect illumination

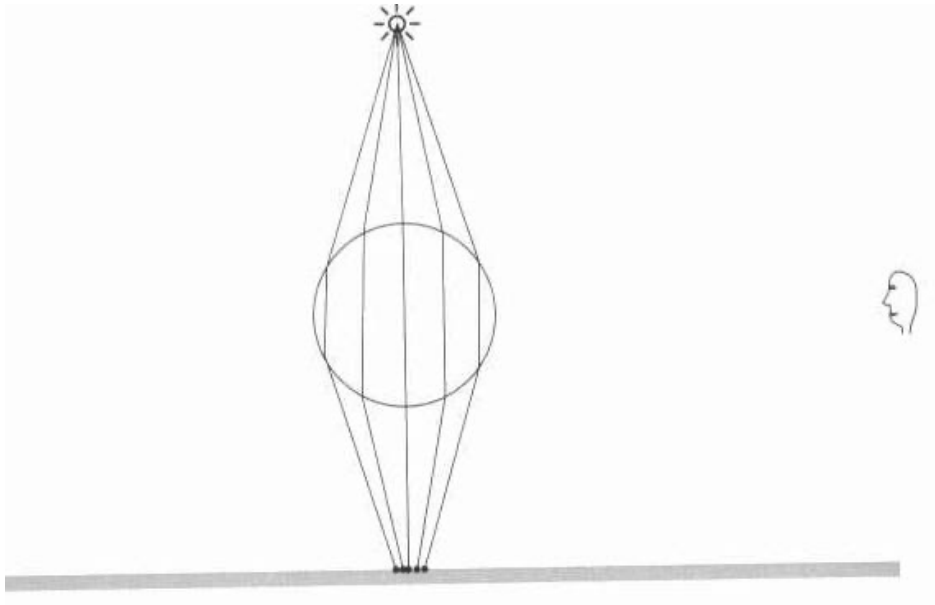
- 10000 photons / 500 photons in radiance estimate



Global Illumination

- global photon map vs. caustics photon map





Photon tracing

- Photon emission
- Photon scattering
- Photon storing

Photon emission

- Given Φ Watt lightbulb.
- Emit N photons.
- Each photon has the power Φ/N Watt
- Photon power depends on the number of emitted photons. Not on the number of photons in the photon map.

What is a photon?

- Flux (power) - not radiance!
- Collection of physical photons
  - A fraction of the light source power
  - Several wavelengths combined into one entity

Diffuse point light

- Generate random direction
- Emit photon in that direction

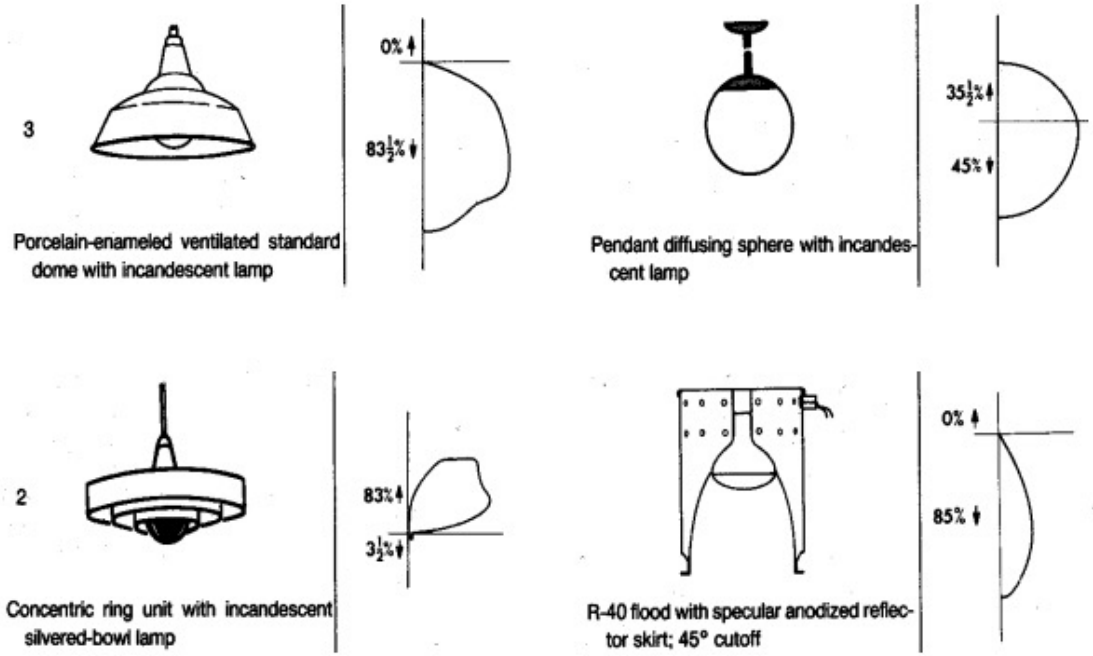
```
// Find random direction
do {
  x = 2.0*random()-1.0;
  y = 2.0*random()-1.0;
  z = 2.0*random()-1.0;
} while ( (x*x + y*y + z*z) > 1.0 );
```

Example: Diffuse square light

- Generate random position p on square
- Generate diffuse direction d
- Emit photon from p in direction d

```
// Generate diffuse direction
u = random();
v = 2*π*random();
d = vector(cos(v)√u,sin(v)√u,√(1−u));
```

Other Sources



- Random directions and importance sampling (e.g., diffuse) within volumes of emission (bounding volumes)
- Projection Maps – Cells in scene which indicate if photons map or not

Surface interactions

- The photon is
  - Stored (at diffuse surfaces) and
  - Absorbed (A) or
  - Reflected (R) or
  - Transmitted (T)

$$A + R + T = 1.0$$

Photon scattering

- The simple way:
  - Given incoming photon with power Φ_p
  - Reflect photon with the power $R*\Phi_p$
  - Transmit photon with the power $T*\Phi_p$

Photon scattering

- The simple way:
  - Given incoming photon with power Φ_p
  - Reflect photon with the power $R*\Phi_p$
  - Transmit photon with the power $T*\Phi_p$
- Risk: Too many low-powered photons - wasteful!
- When do we stop (systematic bias)?
- Photons with similar power is a good thing.

Russian Roulette

- Probability of termination: q

$$E\{X\} = q \cdot 0 + (1-q)\frac{E\{X\}}{1-q} = E\{X\}$$

- Terminate un-important photons and still get the correct result

Russian Roulette Example

- Surface reflectance: $R = 0.5$
- Incoming photon: $\Phi_p = 2W$
-

```
r = random();
if ( r < 0.5 )
  reflect photon with power 2 W
else
  photon is absorbed
```

Russian Roulette Intuition

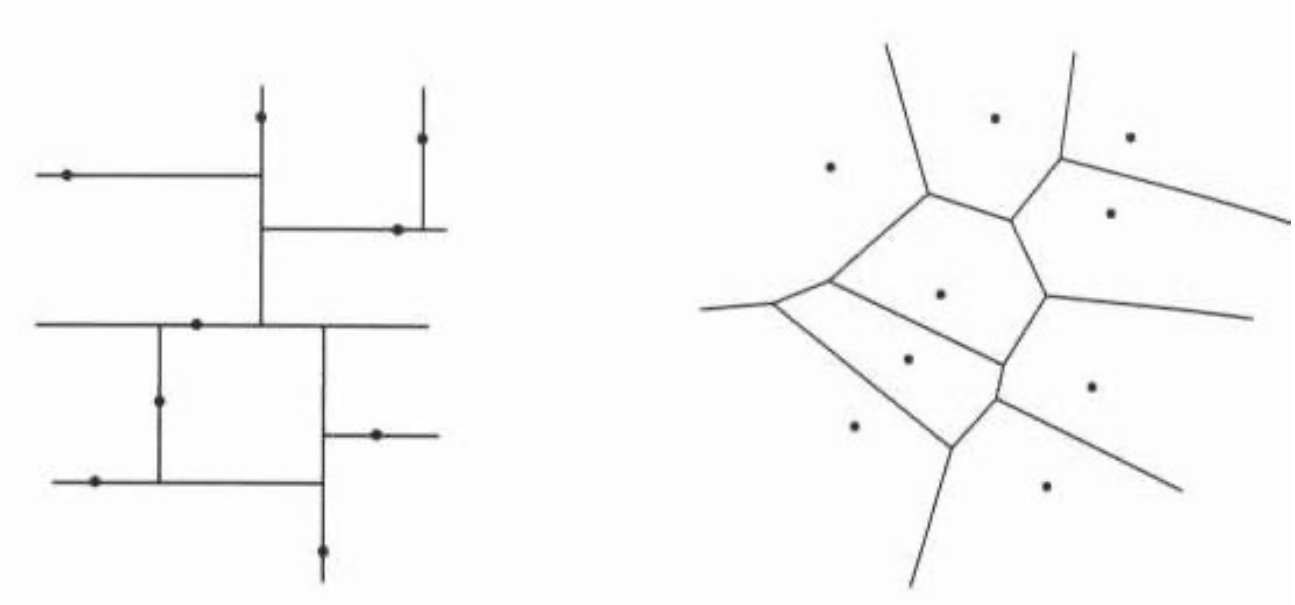
- Surface reflectance: $R = 0.5$
- 200 incoming photons with power: $\Phi_p = 2W$
- Reflect 100 photons with power 2W
- instead of 200 photons with power 1W

Russian Roulette

- Very important!
- Use to eliminate un-important photons
- Gives photons with similar power

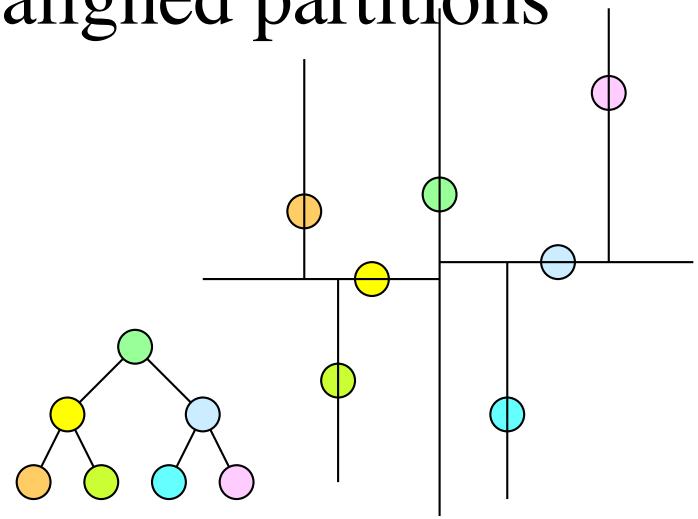
Storing Photons

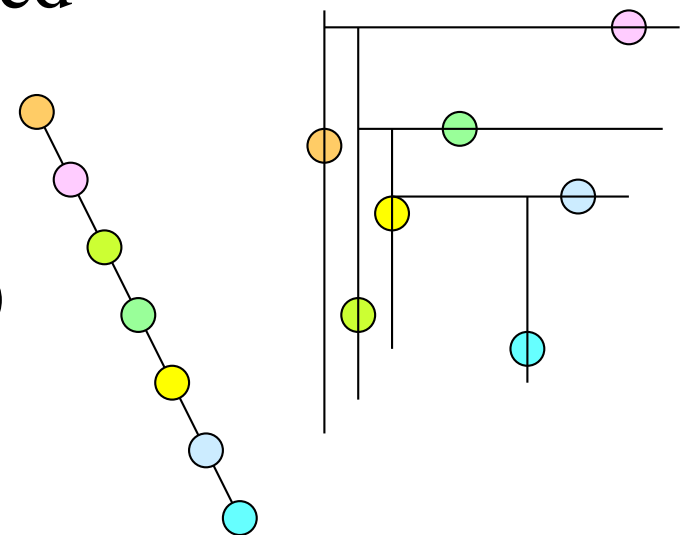
- data structure for storing and searching
- For uniform distribution – cubes are best
- For non-uniform – kd-trees, Voronoi etc.
- kd-trees, perhaps best



Storing Photons

- Use kd-tree – a sequence of axis-aligned partitions
  - 2-D partitions are lines
  - 3-D partitions are planes
- Axis of partitions alternates with depth of the tree
- Average access time - O(log n)
- Worst case O(n) when tree is skewed
- Need to maintain a balanced tree
- Balancing O(n log n)
- k nearest neighbors in O(k + log n)





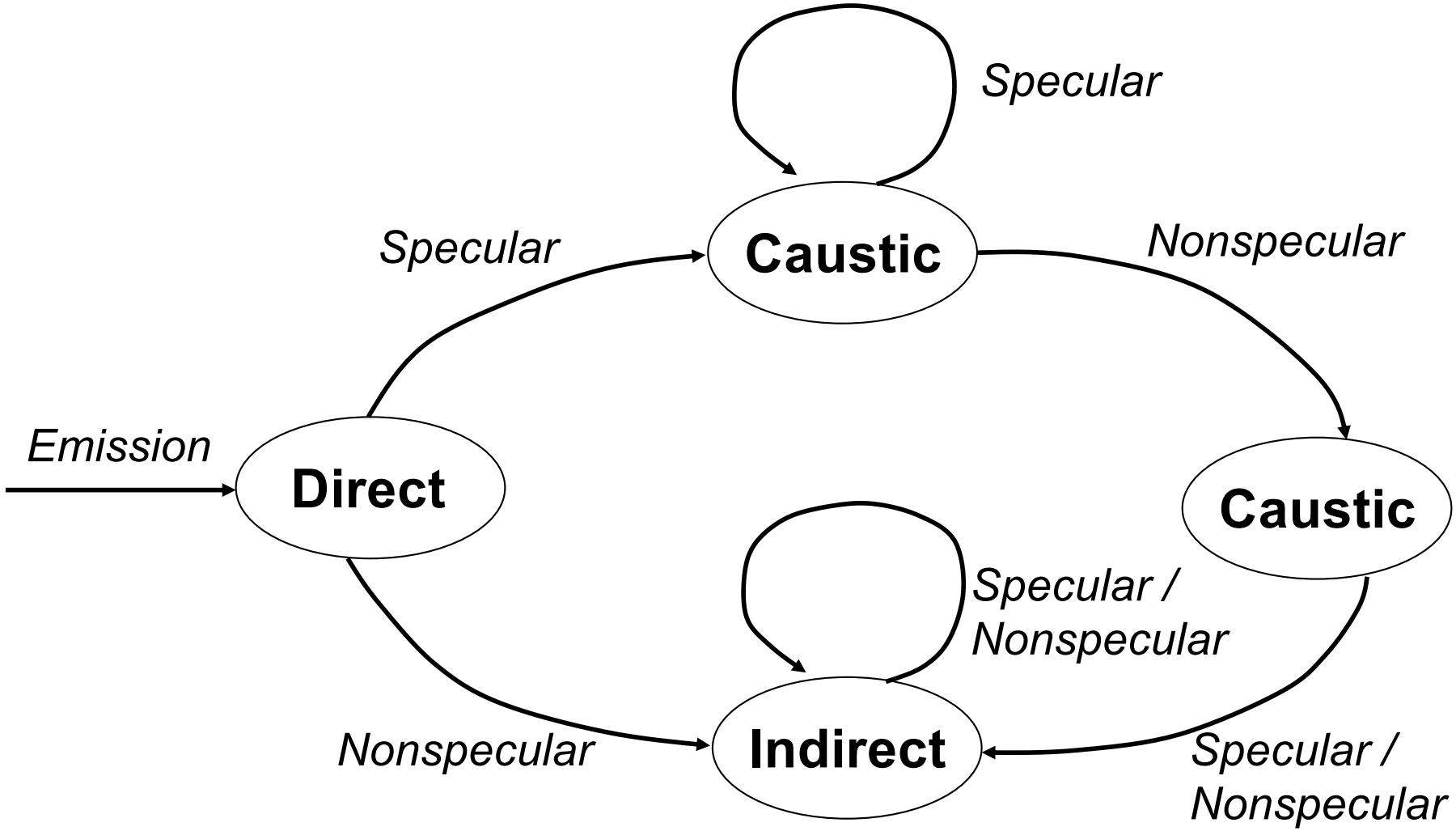
Balancing kd-tree

```
kdtree *balance( points ) {
  Find the cube surrounding the points
  Select dimension dim in which the cube is largest
  Find median of the points in dim
  s1 = all points below median
  s2 = all points above median
  node = median
  node.left = balance( s1 )
  node.right = balance( s2 )
  return node
}
```

Multiple Photon Maps

- Interpolation from the map causes blurriness
- For specular interactions Whitted Ray-Tracing
- Otherwise divide photon map into
  - direct light map $L_{i,d}$
  - Indirect light map $L_{i,i}$
  - Caustic map $L_{i,c}$

Multiple Photon Maps



Photon Mapping: Rendering

- Classification of photons in Photon Map

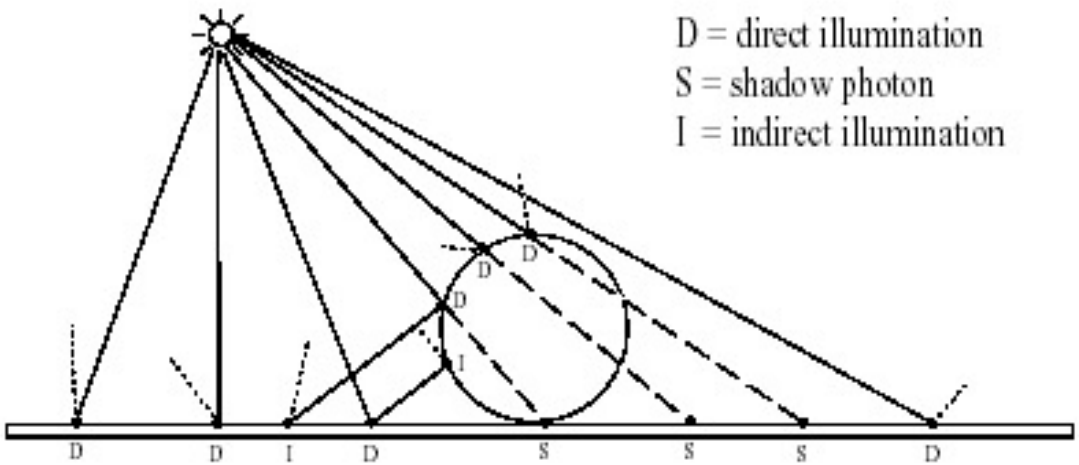


Figure 1: The photons in the global photon map are classified to optimize the rendering of shadows

Multiple Photon Maps

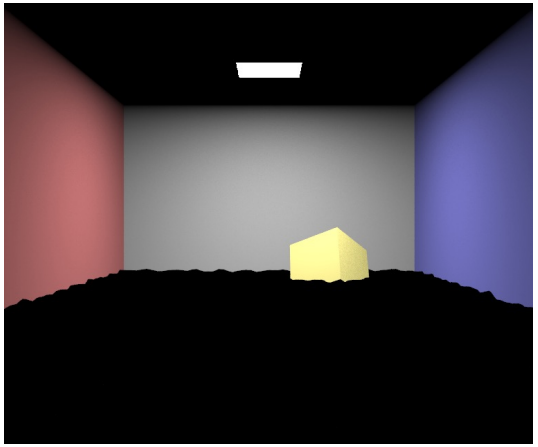
- Direct light map $L_{i,d}$
- Indirect light map $L_{i,i}$
- Caustic map $L_{i,c}$

$$L_o(p,\omega_o) = \int_{S^2} f_r(p,\omega_i,\omega_o) L_i(p,\omega_i) \left|\cos\theta_i\right| d\omega_i$$

$$= \int_{S^2} f_\Delta(p,\omega_i,\omega_o) L_i(p,\omega_i) \left|\cos\theta_i\right| d\omega_i$$

$$+ \int_{S^2} f_{\overline{\Delta}}(p,\omega_i,\omega_o) \left(L_{i,d}(p,\omega_i) + L_{i,i}(p,\omega_i) + L_{i,c}(p,\omega_i) \right) \left|\cos\theta_i\right| d\omega_i$$

Another example



Direct Illumination

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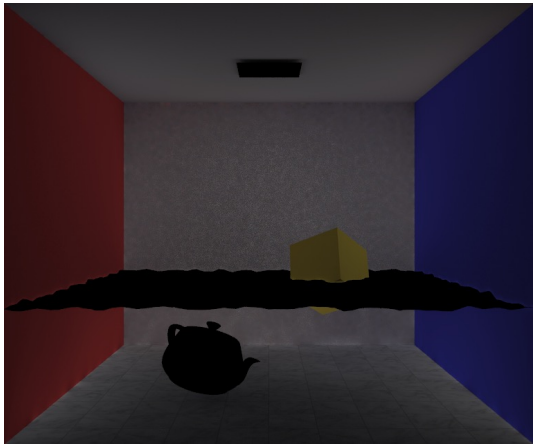

Indirect Illumination

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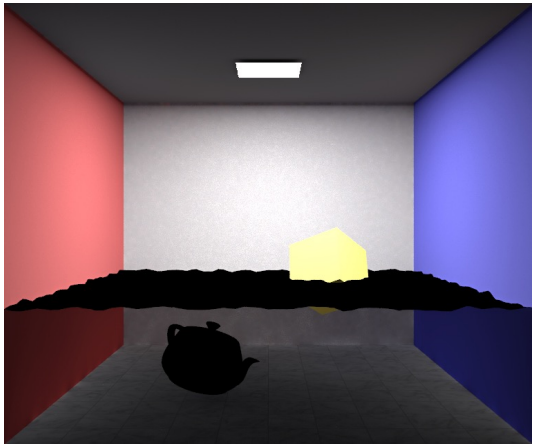

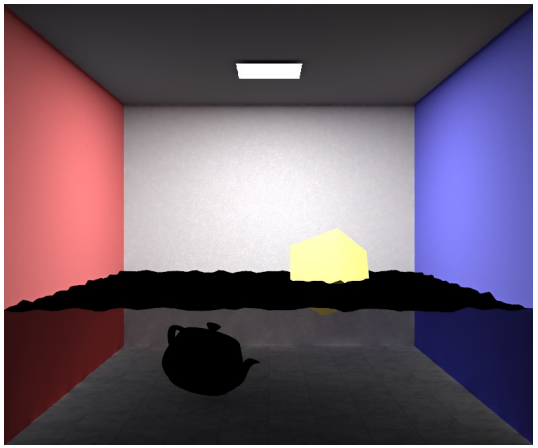

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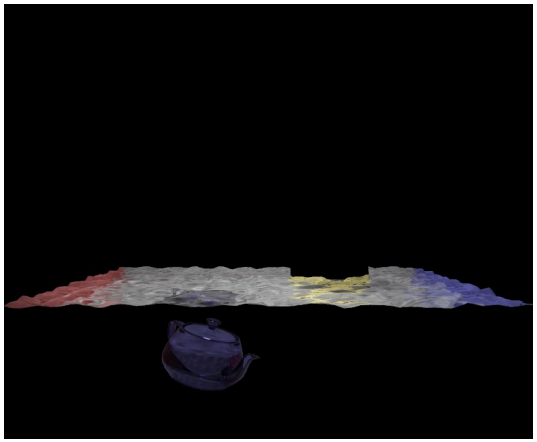

Specular Part

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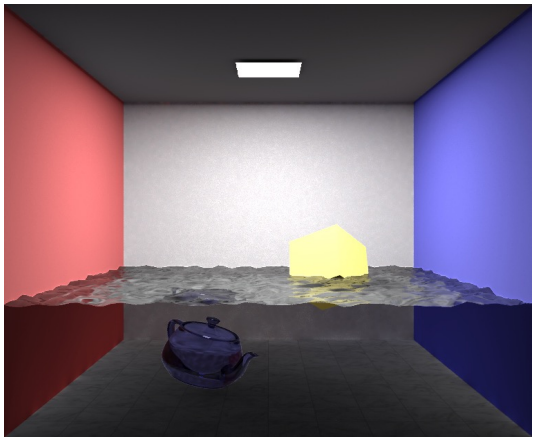
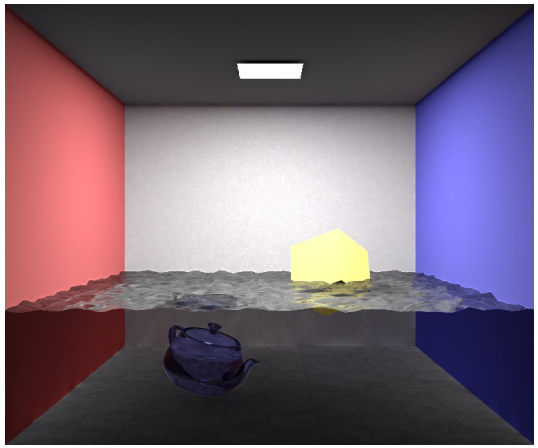


Another example (cont' d)



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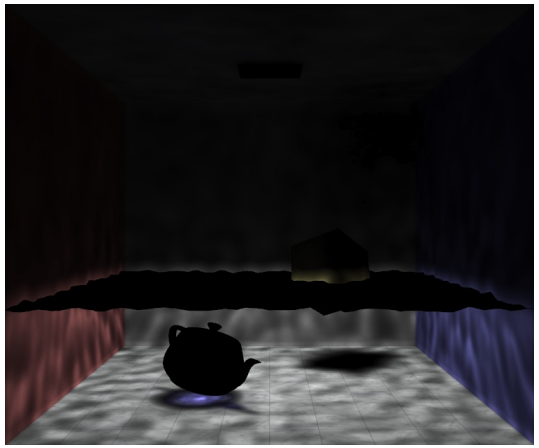

Caustics

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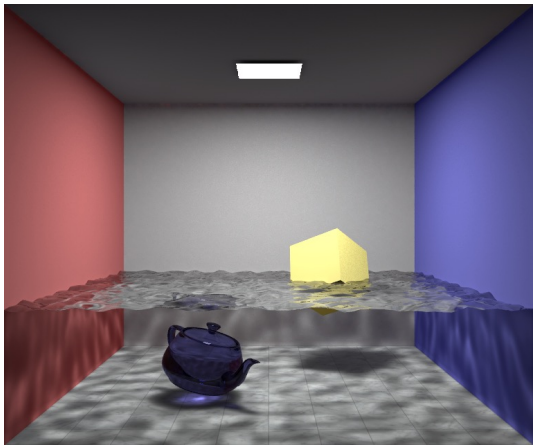
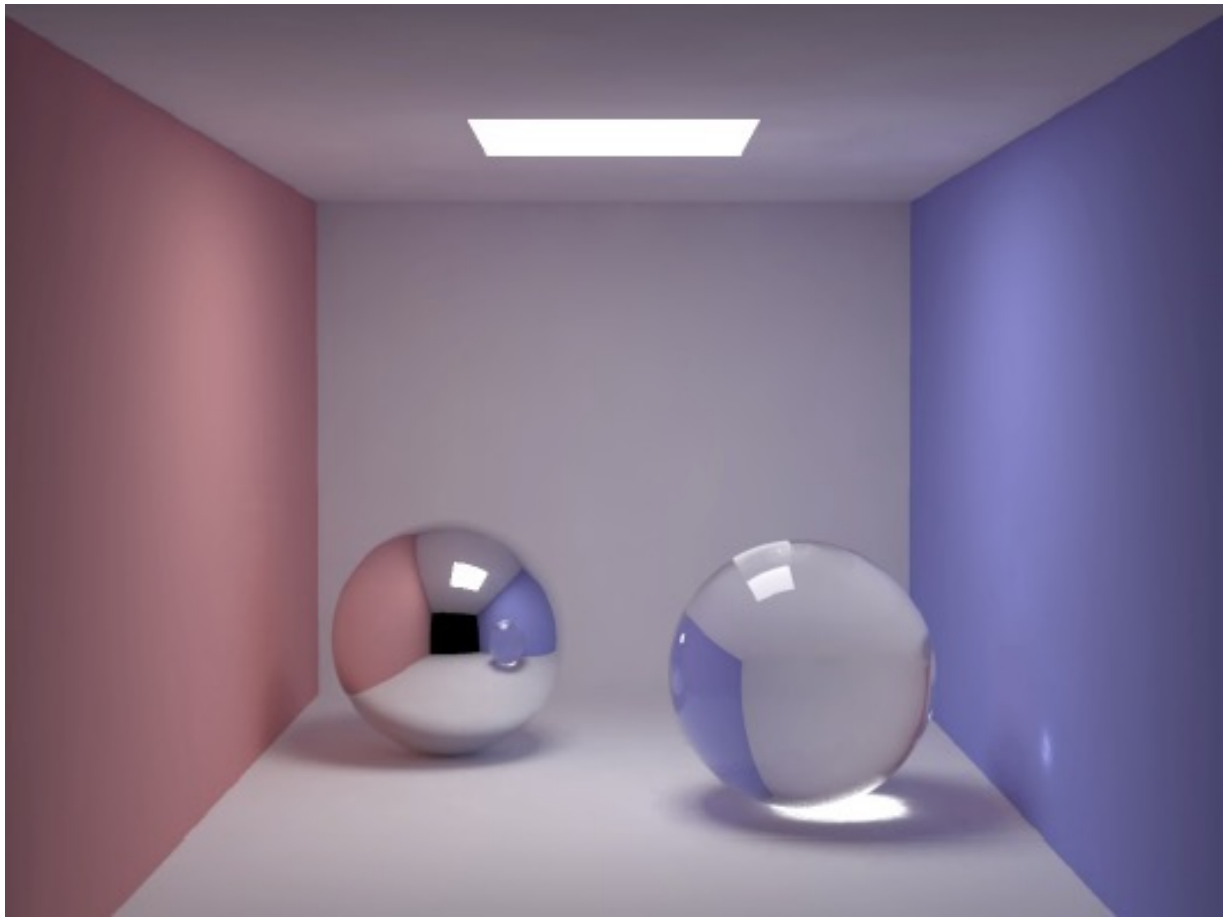


Features

- Photon tracing is unbiased
  - Radiance estimate is biased but consistent
  - The reconstruction error is local
- Illumination representation is decoupled from the geometry

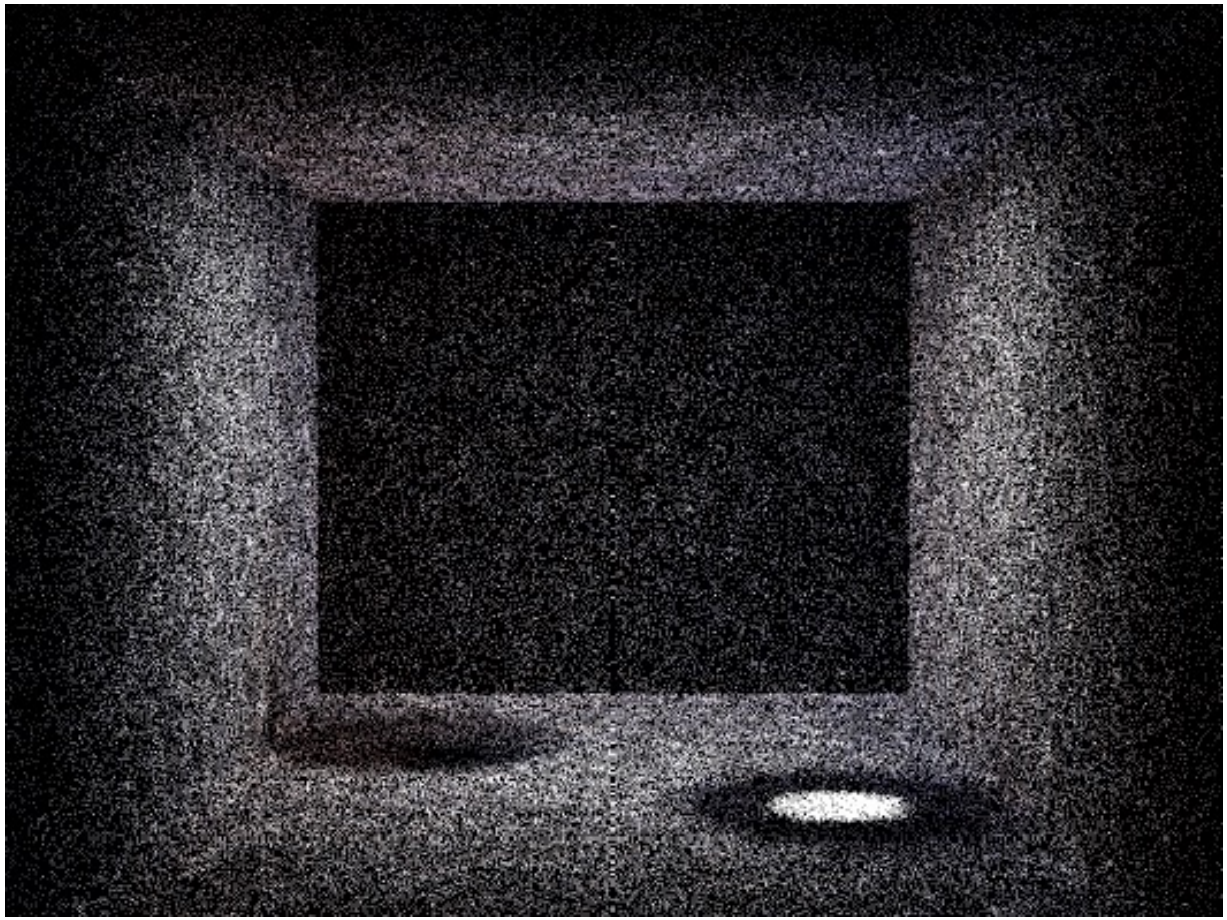
Box

- 200000 global photons, 50000 caustic photons



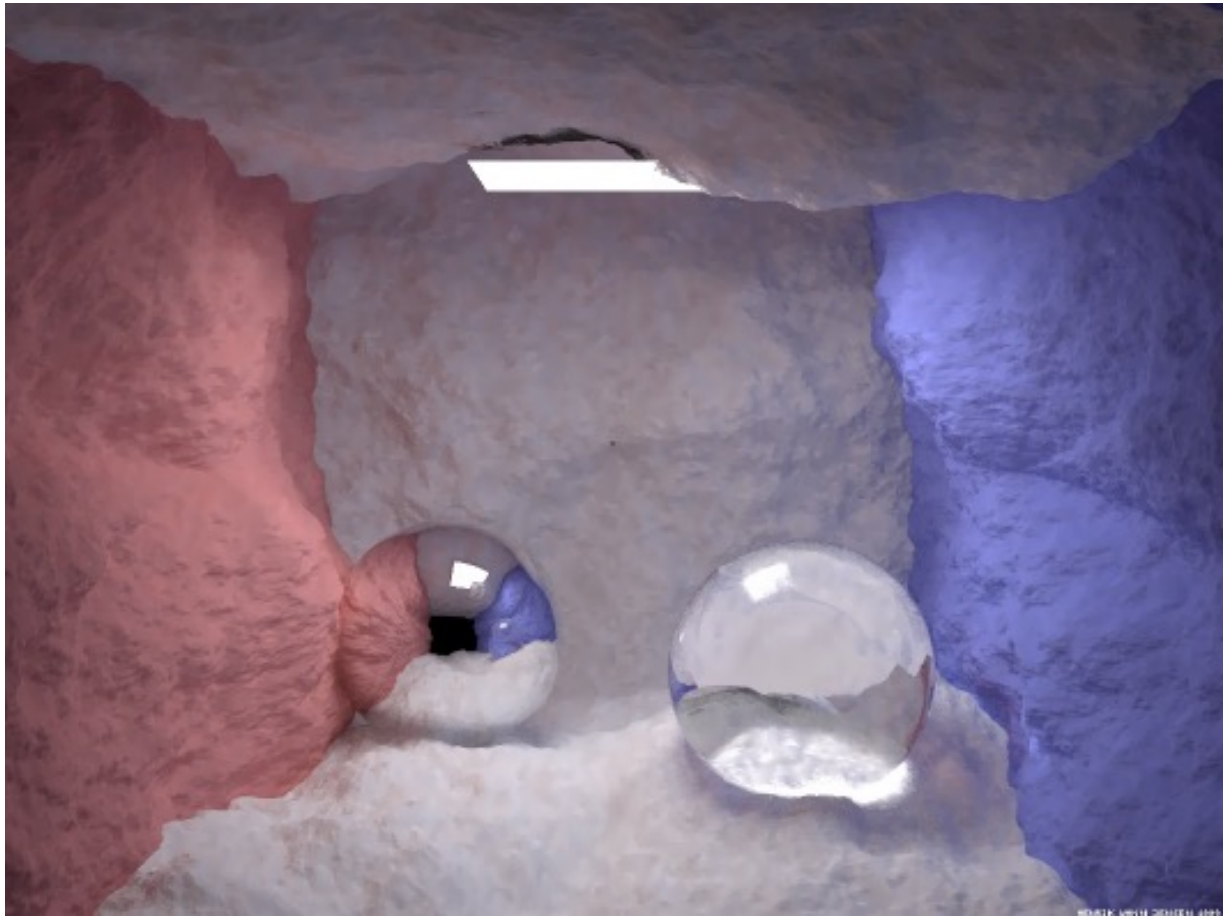
Box: Global Photons

- 200000 global photons

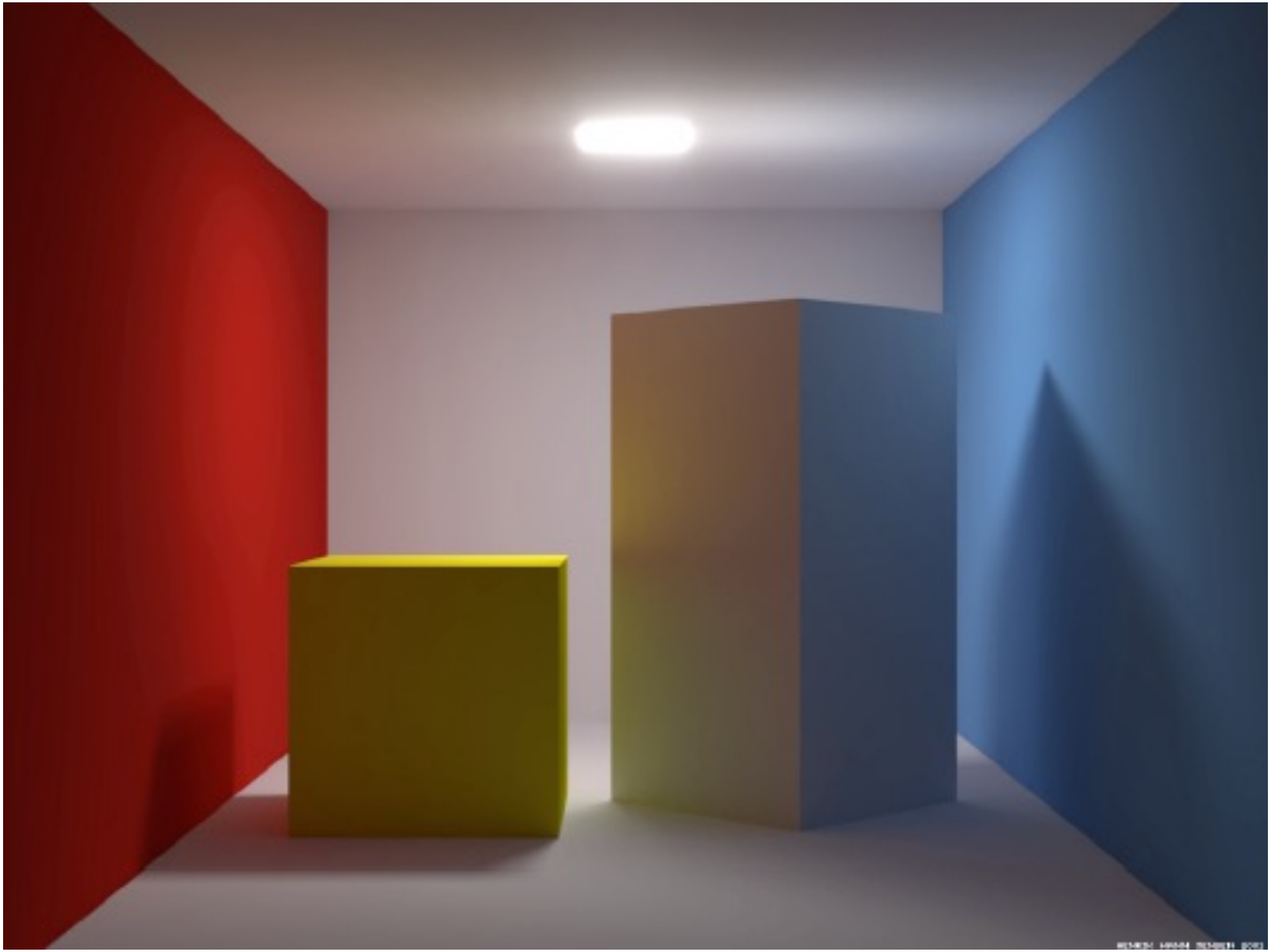


Fractal Box

- 200000 global photons, 50000 caustic photons

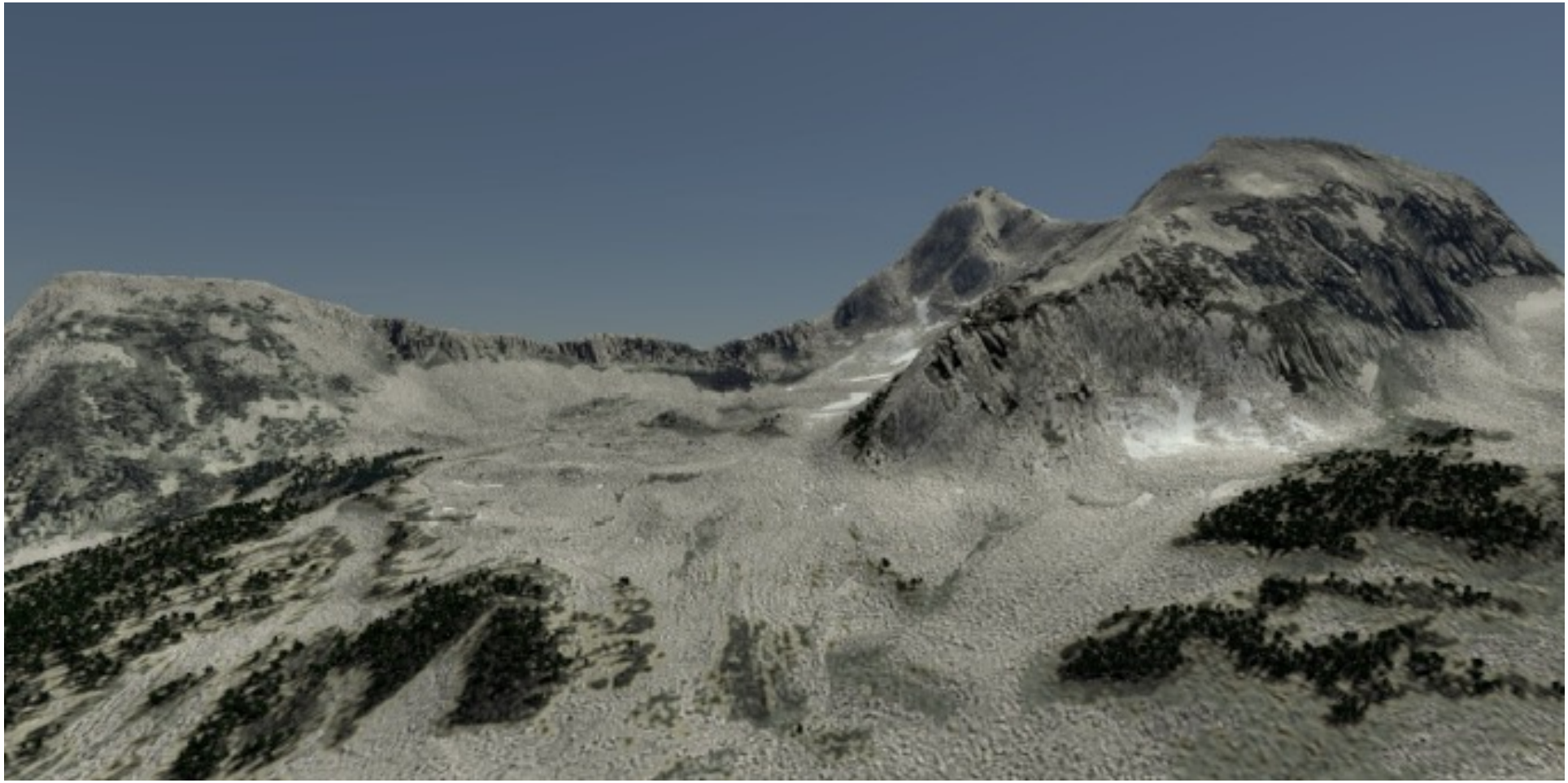


Cornell Box



Indirect Illumination

Little Matterhorn



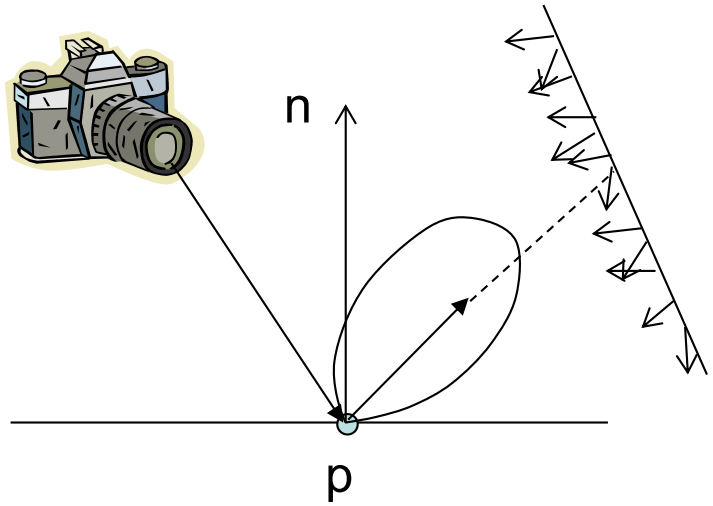
Mies house (swimmingpool)

Mies house (3pm)

Mies house (6pm)

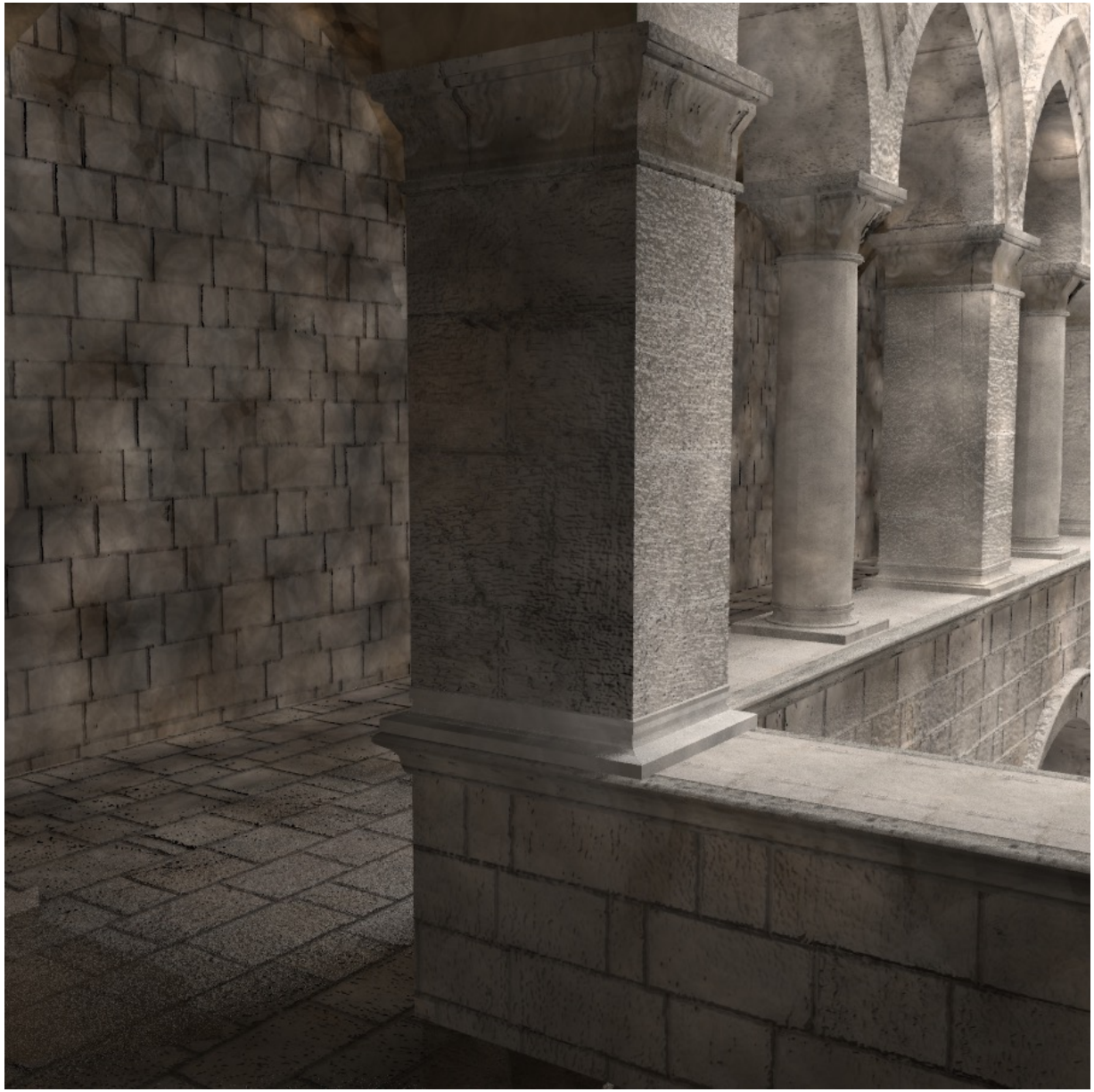
Final Gathering

- Used for higher quality indirect light
- Use BSDF-based distribution of rays at first hit-point in the scene in order to gather the light from direct, indirect and caustic photon maps of places intersected by these rays!

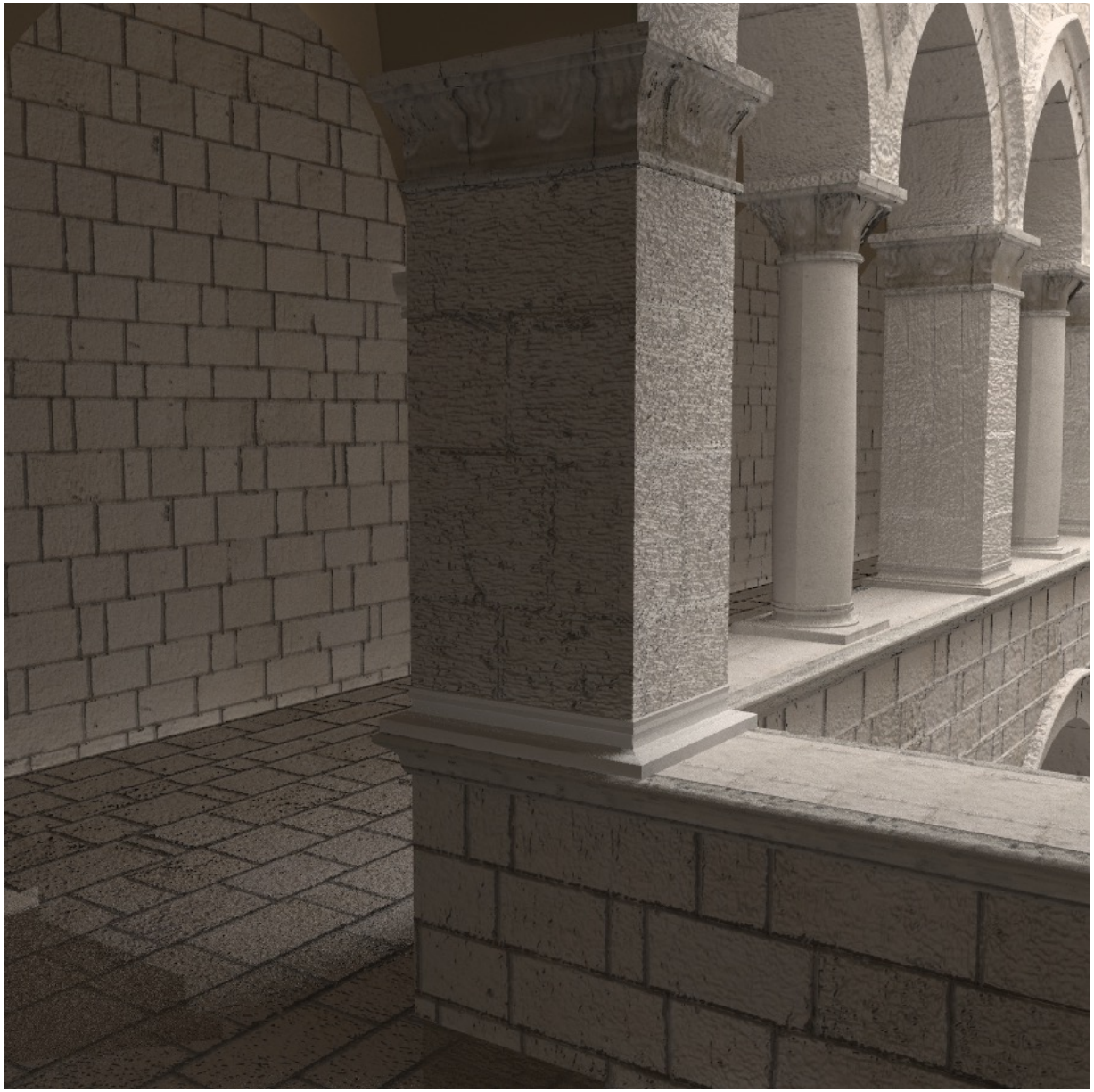


Direct Light only

Photon Map = no Final Gathering



Photon Map = Final Gathering



Recap

- Decouple representation of illumination from geometry
- Photon Map –
  - Illumination as points in a global data structure
  - Cache of light paths
- Estimate illumination based on density
  - Error is of low frequency
  - Not high frequency noise
- Biased method! Average expected value is not correct!
- Consistent – technique will converge as more points/photons are added
- Photon Mapping, Photon Tracing, Photon Map