Outline

1. Quantitative Representations of Uncertainty
   
   partially abstract, top-down
   mainly based on:  Zhenyuan Wang, George J. Klir
                    Generalized Measure Theory, Springer 2009

2. Probabilistic Modeling of Uncertain Fields
   
   mostly concrete, bottom-up
   mainly based on:  recent work at ZIB with
                    Kai Pöthkow, Christoph Petz, Britta Weber
Part 1: Quantitative Representations of Uncertainty

Why bother at all?

- Consideration and quantification of uncertainties is of great importance in many practical applications.
- Vis & VA: part of the data analysis chain + support decision taking.
- Thus we need to understand the data – including their shortcomings, value, relevance, which largely depend on presence/absence of uncertainties.

➡️ We need to
- understand *quantified uncertainty* and deal with it
- perform *uncertainty quantification* by ourselves
What is Uncertainty?

uncertainty $\Leftrightarrow$ lack of information

- Uncertainty due to randomness
  - aleatoric uncertainty
  - Results by chance
  - Lack of information is **objective**
  - *Example*: daily quantity of rain in Seattle

- Uncertainty due to lack of knowledge
  - epistemic uncertainty
  - In principle we could know, but in practice we don’t know
  - Lack of knowledge is **subjective**
  - *Example*: birth date of last Chinese Emperor

Uncertain Propositions - Examples

- “The value of $x$ is between 0.1 and 0.3”

- “The value of $x$ is normally distributed with zero mean and standard deviation 5.0”

- “The value of $x$ is normally distributed”

- “Bob is middle-aged”
Insertion: Fuzzy Sets

- Let \( X \) be a nonempty set \( (X = \text{the “universe” of discourse}) \)

  Fuzzy set in \( X \) characterized by membership function \( m : X \rightarrow [0, 1] \)

- Example: fuzzy set \( X = [-3, 3] \subset \mathbb{R} \)
  \[ m(x) = e^{-x^2} \]

- Example: (crisp) set \( X = [-3, 3] \subset \mathbb{R} \)
  \[ m(x) = 1 \]

Example: Modelling of Imprecise Age Statements

- Measure in years; age interval of human beings: \( X = [0, 100] \subset \mathbb{N} \)

  young
  \[ m(x) = \begin{cases} 1 & \text{if } x \leq 25 \\ \frac{40-x}{15} & \text{if } 25 < x < 40 \\ 0 & \text{if } x \geq 40 \end{cases} \]

  not young
  \[ m(x) = \begin{cases} 0 & \text{if } x \leq 25 \\ \frac{x-25}{15} & \text{if } 25 < x < 40 \\ 1 & \text{if } x \geq 40 \end{cases} \]

  old
  \[ m(x) = \begin{cases} 0 & \text{if } x \leq 25 \\ \frac{x-25}{15} & \text{if } 25 < x < 40 \\ 1 & \text{if } x \geq 40 \end{cases} \]

  not old
  \[ m(x) = \begin{cases} 0 & \text{if } x \leq 25 \\ \frac{x-25}{15} & \text{if } 25 < x < 40 \\ 1 & \text{if } x \geq 40 \end{cases} \]
Example: Modelling of Imprecise Age Statements

- Age interval of human beings: \( X = [0, 100] \subset \mathbb{N} \)

![Graph showing age distribution](image)

\[
m(x) =
\begin{cases}
0 & \text{if } x \leq 25 \\
\frac{x-25}{15} & \text{if } 25 < x < 40 \\
1 & \text{if } x \geq 40 \\
\frac{65-x}{15} & \text{if } 50 < x < 65 \\
0 & \text{if } x \geq 65 
\end{cases}
\]

Puzzles and Problems

**Paradoxon of total ignorance**

Is there life beyond Earth?

Case 1: beyond Earth: life | no life
---|---
Ignorant’s response: \( \frac{1}{2} \) | \( \frac{1}{2} \)

Case 2: animal life | plant life | no life
---|---|---
Ignorant’s response: \( \frac{1}{3} \) | \( \frac{1}{3} \) | \( \frac{1}{3} \)

Answers inconsistent: from case 2: \( P(\text{life}) = \frac{2}{3} > \frac{1}{2} = P(\text{no life}) \)
from case 1: \( P(\text{animal life}) = \frac{1}{4} < \frac{1}{3} = P(\text{no life}) \)

⇒ Uniform probabilities on distinct representations of the state space are inconsistent.
⇒ A probability distribution cannot model ignorance (maximal incompleteness).
Puzzles and Problems

*Imprecise* measurement with digital outcome:

- *Observations* at the boundaries of the intervals are unreliable → they should be properly discounted

- Taking measurements for union of the 2 events → one of the discount rate peaks is not applicable
  
  - *same observations* produce *more evidence* for single event \(0 \cup 1\) then for 2 disjoint events \(0, 1\)

  \[ \text{prob}(0 \cup 1) \geq \text{prob}(0) + \text{prob}(1) \]  

  *non-additive* !

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Mathematical Modelling of Uncertainty

- A variety of *types of uncertainty* occur in practice, including *mixtures*.

- Quantification of uncertainties, including mixtures, requires a unifying mathematical framework.

- Establishing such a mathematical framework is *difficult*! (it already required centuries …)

- Development of such a theory is not yet fully accomplished, but silhouettes start to become visible!
What I will outline here

- What is the overall picture?
- What are the major types of modeling?
- What is the general mathematical framework behind?
- Where can I find further information?

And what not:

- Any technical details about the theories
- Illustrating examples

Fundamental Setting

- $X$ : set of all elementary events (= the “universe”)
- Situation with possible outcomes or occurrences of “events” $A, B, C, ...$
- Events $A, B, C, ...$ are subsets of $X$, i.e. elements of power set of $X$
  
  ➔ Tasks:
  
  Measure the
  evidence that event $A$ happened
  degree of truth of the statement “event $A$ happened”
  probability that event $A$ will happen
Measures in Mathematics

to measure = to assign real numbers to sets

• Classical task in metric geometry: assign numbers to geometric objects for lengths, areas, or volumes

• Requirement: assigned numbers should be \textit{invariant} under displacement of respective objects

• In ancient times: to measure = to compare with a standard unit

Measure in Mathematics

• Soon: \textit{problem of incommensurables}

\[
\begin{array}{c}
1 \\
\hline \\
1 \\
\end{array} \quad \begin{array}{c}
? \\
\hline \\
?
\end{array} \quad \pi \approx 3.141592653589793
\]

⇒ Measurement is more complicated than initially thought. It involves infinite processes and sets.

• 1854: First tool to deal with the problem: \textit{Riemann integral}

⇒ Enables to compute lengths, areas, volumes for complex shapes (as well as other measures).
Measures in Mathematics

• ~ 1870s and 80s: Riemann integral has a number of deficiencies
  • Applicable only to functions with finite number of discontinuities
  • Fundamental operations of differentiation and integration are in general not reversible in the context of Riemann theory
  • Limit processes can in general not be interchanged:
    \[ \int_a^b \lim_{n \to \infty} f_n(x) \, dx \quad \text{and} \quad \lim_{n \to \infty} \int_a^b f_n(x) \, dx \]
    may differ.

Measures in Mathematics

• 1898: Émile Borel developed classical measure theory
  • Defined σ-algebra = class of sets that is closed under set union of countably many sets and set complement
  • Defined measure \( \mu \) that associates a number \( \in \mathbb{R}^+ \) with each bounded subset in the σ-algebra
  • The measure is additive:
    \[ \mu(A + B) = \mu(A) + \mu(B) \quad \text{if} \quad A \cap B = \emptyset \]
Measures in Mathematics

- 1899-1902: Henry Lebesgue defined integral
  - Based on a measure that subsumes the Borel measure as a special case
  - Connected measures of sets and measures of functions

- 1933: Andrey Nikolaevich Kolmogorov developed the concept of probability measure
  - Used classical measure
  - Added: measure 1 is assigned to the universal set

→ Classical Probability Theory

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Measures in Mathematics

- About 50 years later: additivity requirement became a subject of controversy
  - Too restrictive to capture e.g. the full scope of measurement:
    - Works well under idealized error-free measurements
    - Not adequate when measurement errors are unavoidable

- The two basic types of uncertainties in relation to experiments:
  - Aleatoric: results differ each time she/he runs an experiment
    - phenomenon is truly random; results „depend“ on chance
    → probabilistic modeling
  - Epistemic: in principle we could know the exact results, but we don’t know in practice;
    - due to errors that practically cannot controlled;
    → non-probabilistic modeling
Measures in Mathematics

1954 Gustave Choquet developed a (potentially infinite) family of non-additive measures (“capacities”)

- For each given capacity there exists a dual “alternating capacity”
- Integral based on these measures (Choquet integral)
  - non-additive
  - can be computed using Riemann or Lebesgue integration
  - applied specifically to membership functions and capacities

Dempster-Shafer Theory

Motivation: precision required in classical probability not realistic in many applications

1967 Arthur P. Dempster introduced imprecise probabilities

- Deal with convex sets of probability measures rather than single measures
- For each given convex set of probability measures he introduced
  - 2 types of non-additive measures: lower & upper probabilities
    super- & supra-additive

Allows to represent probabilities imprecisely by intervals of real numbers.
**Dempster-Shafer Theory**

1976 Glenn Shafer analyzed special types of lower & upper probabilities called them belief & plausibility measures.

- Theory based on these measures = Dempster-Shafer theory (DST) or evidence theory.

- DST is capable of dealing with interval-based probabilities: [belief measure, plausibility measure] = ranges of admissible probabilities.

- Turns out: belief measures = Choquet capacities of order $\infty$; plausibility measures = alternating capacities of order $\infty$.

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**Measures in Mathematics**

1978 Michio Sugeno tried to compare

- membership functions of fuzzy sets
- with probabilities

not directly possible

- Generalization of additive measure analogous to generalization:
  
<table>
<thead>
<tr>
<th>crisp sets</th>
<th>fuzzy sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>additive measure</td>
<td>fuzzy measure</td>
</tr>
<tr>
<td>monotone measure</td>
<td></td>
</tr>
</tbody>
</table>

- Introduced also Sugeno integral with respect to a monotone measure.
Measures in Mathematics

1978 Lotfi Zadeh defined:

- "Possibility function" associated with each fuzzy set (numerically: membership function)
- "Possibility measure" supremum of the possibility function in each set of concern (both for crisp and fuzzy sets)

- One of several interpretations of the "theory of graded possibilities"
- Connection to DST:
  - plausibility measures = possibility measures (consonant plausibility measures)
  - belief measures = necessity measures (consonant belief measures)

Classes of Uncertainty Theories

Using additive measures
- $\mu(A \cup B) = \mu(A) + \mu(B)$ expresses no interaction between events
  - classical probability + measure theory

Using non-additive measures
- $\mu(A \cup B) > \mu(A) + \mu(B)$ expresses positive interaction between events
  - synergy, cooperation, coalition, enhancement, amplification
- $\mu(A \cup B) < \mu(A) + \mu(B)$ expresses negative interaction between events
  - incompatibility, rivalry, inhibition, downgrading, condensation
  - (many) uncertainty theories + generalized measure theory
Most Utilized Uncertainty Theories + Further Reading

1. Classical Probability Theory

2. Dempster-Shafer Theory

Simona Salicone:
"Measurement Uncertainty: An Approach via the Mathematical Theory of Evidence",
Springer, 2007

Jürg Kohlas, Paul-Andre Monney:
Springer, 1995

3. Possibility Theory

Didier Dubois and Henri Prade:
"Possibility Theory, Probability Theory and Multiple-valued Logics: A Clarification",

Gerla Giangiacomo:
"Fuzzy logic: Mathematical Tools for Approximate Reasoning",

Thank you very much for your attention!
Part 2: Probabilistic Modeling of Uncertain Fields

Sources:

Probabilistic marching cubes.
Kai Pöthkow, Britta Weber, HCH

Probabilistic local features in uncertain vector fields with spatial correlation
Christoph Petz, Kai Pöthkow, HCH

Approximate level-crossing probabilities for interactive visualization of uncertain isocontours.
Kai Pöthkow, Christoph Petz, HCH
*Int. J. Uncertainty Quantification* (2012; forthcoming)

Uncertain Scalar Field

Model as *discrete random field*
For simplicity: use *Gaussian random variables*

Each field configuration: conceived as a realization of a multivariate Gaussian RV
Gaussian Random Field

Discrete random field = multivariate Gaussian RV

\[ Y \sim \mathcal{N}_n(\mu, \Sigma) \]
\[ \mu = [E(Y_1), E(Y_2), \ldots, E(Y_n)] \]
\[ \Sigma = [\text{Cov}(Y_i, Y_j)]_{i=1,2,\ldots,n; j=1,2,\ldots,n} \]

\[ Y(y) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp \left( -\frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu) \right) \]

Gaussian Random Field

Compute probability of locally defined events, e.g.

\[ \text{prob}(x_1 \in [a_1, b_1] \text{ and } x_2 \in [a_2, b_2]) \]

- Sum over all configurations that respect to predicate in the argument of \text{prob}(\ldots)
- Majority of variables are “integrated out” (marginalized)
- Then only a few local integrations remain
Gaussian Random Field

Marginalization:

\[
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dy_{m+1} \ldots dy_n \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp \left( -\frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu) \right)
\]

\[
= \frac{1}{(2\pi)^{m/2} \det(\Sigma')^{1/2}} \exp \left( -\frac{1}{2} (\tilde{y} - \tilde{\mu})^T \tilde{\Sigma}^{-1} (\tilde{y} - \tilde{\mu}) \right)
\]

\[
=: f_{\tilde{\nu}}(y_1, \ldots, y_m)
\]

where \( \tilde{Y} \) is the reduced random vector and \( \tilde{y}, \tilde{\mu} \) and \( \tilde{\Sigma} \) are the quantities \( y, \mu \) and \( \Sigma \) with \( n - m \) columns/rows deleted corresponding to the marginalized variables \( y_{m+1} \ldots y_n \)
Probabilities of Classes of Realizations

Constrain \( m < n \) RV \( Y_i \) to subsets \( S_i \).

Re-order RV such that constrained ones are the first \( m \) ones.

Probability of constrained realization:

\[
\text{Prob} (Y_1 \in S_1, \ldots, Y_m \in S_m) = \int_{S_1} \cdots \int_{S_m} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_X(y_1, \ldots, y_n)
\]

For Gaussian distribution:

\[
\int_{S_1} \cdots \int_{S_m} f_{\mathcal{N}}(y_1, \ldots, y_m)
\]

Level Crossing Probabilities

For any realization (= grid function \( g \)) assume an \( C^0 \) interpolant taking its extreme values at the sample points.

Consider a particular grid cell \( c \) with vertex indices \( \bar{I} \in I \).

Cell \( c \) crosses \( \vartheta \)-level of \( g(y) \) iff not all differences \( (y_i - \vartheta)_{i \in \bar{I}} \) have the same sign.

Level crossing probability \( \text{Prob}_c(\vartheta\text{-crossing}) \):

Integrate \( \{Y_i\}_{i \in \bar{I}} \) over sets \( \{y_j \in \mathbb{R} \mid y_j \geq \vartheta\} \) and \( \{y_i \in \mathbb{R} \mid y_i \leq \vartheta\} \)
Level Crossing Probabilities on Edges

Edge with bivariate Gaussian RV $Y = [Y_1, Y_2]$

$$\text{Prob}(\theta\text{-crossing}) =$$
$$= \text{Prob}(Y_1 \leq \theta, Y_2 > \theta) + \text{Prob}(Y_1 > \theta, Y_2 \leq \theta)$$
$$= \int_{y_1 \leq \theta} dy_1 \int_{y_2 > \theta} dy_2 f_Y(y_1, y_2, \theta, \mathbf{y}) + \int_{y_1 > \theta} dy_1 \int_{y_2 \leq \theta} dy_2 f_Y(y_1, y_2, \theta, \mathbf{y})$$

Level Crossing Probabilities on Faces

4 Cases (after Symmetry Reduction)

\begin{align*}
\text{Corresponding Integrals} & \quad \triangleleft \\
\end{align*}

| \text{Case} | P_{\theta,1} & P_{\theta,2} & P_{\theta,3} & P_{\theta,4} \\
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$\int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)$</td>
<td>$\int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)$</td>
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<tr>
<td>2</td>
<td>$\int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)$</td>
<td>$\int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)$</td>
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</tr>
<tr>
<td>3</td>
<td>$\int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)$</td>
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</table>
Level Crossing Probabilities on Rectangular Cells, …

Types of integrals \( \Delta \) symmetry-reduced Marching cubes cases.

In **2D**: 4 distinct cases (1 non-crossing, 3 crossing)

In **3D**: 15 distinct cases (1 non-crossing, 14 crossing)
In **4D**: 223 distinct cases (1 non-crossing, 222 crossing)

In **nD**: use Polya’s counting theory

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Level Crossing Probabilities – Simplified

# of cases (i.e. integrals) **with** level crossings grows with dimension …

**Better** exploit \( \text{Prob}_c(\theta\text{-crossing}) = 1 - \text{Prob}_c(\theta\text{-non-crossing}) \)

only 2 cases **without** level crossings

\( \rightarrow \text{ for all dimensions only 2 integrals!} \)

e.g. for square cells in 2D:

\[
\text{Prob}_c(\theta\text{-crossing}) = 1 - \int dy_1 \int dy_2 \int dy_3 \int dy_4 f_Y(y_1, y_2, y_3, y_4)
\]

\[
(y_1 \leq \theta \land y_2 \leq \theta \land y_3 \leq \theta \land y_4 \leq \theta) \lor (y_1 > \theta \land y_2 > \theta \land y_3 > \theta \land y_4 > \theta)
\]

**But** dimension of integrals still = # vertices of geometric object!
Algorithm & Implementation

- Preprocessing
  - Estimate $\hat{\mu}_i$ for all sample points
  - Estimate $\widehat{\text{Cov}}_{i,j}$ for all 2- or 3-cells

- For a given iso-value $\Theta$
  - Estimate crossing probabilities using Monte Carlo integration

Level Crossing Probabilities on Faces
Algorithm & Implementation

for each cell $c$ {
    $L_c \leftarrow$ CholeskyDecomposition($\Sigma_c$)
    #crossings $\leftarrow$ 0
    for 1...#samples {
        $y \leftarrow$ random numbers $y_1 \ldots y_m \sim \mathcal{U}(0, 1)$
        $y \leftarrow$ BoxMullerTransform($y$)
        $y \leftarrow L_c y + \mu_c$
        if(crossing$_c(y))$ #crossings $\leftarrow$ #crossings + 1
    }
    Prob$_c \leftarrow$ #crossings/#samples
}

Impact of Spatial Correlations

$q=0.00$

$q=0.65$

$q=0.95$

synthetic data
Climate Simulation

Data courtesy of ECMWF

\( \hat{\mu}_i \)  \( \hat{\text{Cov}}_{i,j} \)

Isotherm of Climate Simulation

spatial correlations considered

not considered
Fuel Injection Data Set + Artificial Noise: Uncertain Level Set

Application Example: Isotherm of Climate Simulation

spatial correlations not considered
Application Example: Isotherm of Climate Simulation

Local Features in Uncertain Vector Fields

Probabilistic local features in uncertain vector fields with spatial correlation
Christoph Petz, Kai Pöthkow, HCH
Previous Work

Wittenbrink, Pang & Lodha
Glyphs for visualizing uncertainty in vector fields
TVCG, 1996

Friman, Hennemuth, Harloff, Bock, Markl & Peitgen,
Probabilistic 4D blood flow tracking and uncertainty estimation,
Medical Image Analysis, 2011

Otto, Germer, Hege & Theisel
Uncertain 2D Vector Field Topology,
Eurographics 2010

Discretized Vector Fields

crisp vector field

uncertain vector field
Tasks

- Define a model to represent uncertain vector fields considering spatial correlation
- Establish a framework for local probabilistic feature extraction from vector fields
- Estimate probabilities for the existence of critical points and vortex cores

Uncertain Vector Fields

- Again modeled as discrete random field
- For simplicity: Normal distributions \( \mathbf{Y} \sim \mathcal{N}_n(\mu, \Sigma) \)

\[
\mathbf{Y}(\mathbf{y}) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp \left( -\frac{1}{2} (\mathbf{y} - \mu)^T \Sigma^{-1} (\mathbf{y} - \mu) \right)
\]

\[
\mu = [\mathbb{E}(Y_1), \mathbb{E}(Y_2), \ldots, \mathbb{E}(Y_n)]
\]

\[
\Sigma = [\text{Cov}(Y_i, Y_j)]_{i=1,2,\ldots,n; j=1,2,\ldots,n}.
\]
Marginalization

Local features can be identified

- at each cell (and its neighborhood)
- using local marginal distributions

\[ v_c = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma) \]

\[ \widehat{\mu} : C_\eta \to \mathbb{R}^{K_cN} \]
\[ \widehat{\Sigma} : C_\eta \to \mathbb{R}^{K_cN \times K_cN} \]

- Feasible for Gaussian fields only

Probabilistic Feature Extraction

Feature indicator

\[ I : C_\eta \times \mathbb{R}^{K_cN} \to \{0, 1\} \]

Feature probability

\[ P(c) = \int_D f_c(v) \, dv = \int_{\mathbb{R}^{K_cN}} f_c(v) I(c, v) \, dv = \mathbb{E}(I(c, \cdot)) \]

where \[ D = \{ v \in \mathbb{R}^{K_cN} | I(c, v) = 1 \} \]
**Critical Points in 2D**

Compute Poincaré-index (winding number)

\[ \text{idx}(c, v) = \frac{\sum_{i=0}^{K-1} \angle(v_i, v_{(i+1) \% K})}{\sum_{i=0}^{K-1} \theta_i} \]

**Critical Point Classification**

\[ I_{\text{source}}(c, v) = \begin{cases} 1 & \text{idx}(c, v) > 0 \land \text{div}(c, v) > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{\text{sink}}(c, v) = \begin{cases} 1 & \text{idx}(c, v) > 0 \land \text{div}(c, v) < 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{\text{saddle}}(c, v) = \begin{cases} 1 & \text{idx}(c, v) < 0 \\ 0 & \text{otherwise} \end{cases} \]
Critical Points in 3D

- For linear tetrahedral elements: 12-dimensional random vectors have to be considered.
- Compute the Poincaré-index using solid angles.

\[ I_+ (c, v) = \begin{cases} 1 & \text{if } \text{idx}(c, v) > 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ I_- (c, v) = \begin{cases} 1 & \text{if } \text{idx}(c, v) < 0 \\ 0 & \text{otherwise} \end{cases} \]

Vortex Cores

- Indicator for vortices (Sujudi-Haines criterion)
  - Jacobian J has 2 complex eigenvalues
  - Real eigenvector is parallel to the vector field
- J is piecewise constant → vortex cores are locally straight lines
- Compute probability for the existence of a vortex core
1\textsuperscript{st} Computational Step: Empirical Parameter Estimation

- Arithmetic mean

\[ \hat{\mu} = \frac{1}{L} \sum_{i=1}^{L} \tilde{v}_i \]

- Empirical covariance matrix

\[ \hat{\Sigma} = \frac{1}{L-1} \sum_{i=1}^{L} (\tilde{v}_i - \hat{\mu}) (\tilde{v}_i - \hat{\mu})^T \]

2\textsuperscript{nd} Computational Step: Monte-Carlo Integration

- Compute locally correlated realizations

- Estimate feature probability using the ratio of occurrences
Critical-Point Probabilities in Wall-Shear-Stress Fields

WSS = vector field on surface

- Intensity encodes probabilities
- Color encodes type of CP: sinks in violet, sources in green
- and saddles in blue. Intensities are scaled by the probabilities.

Probabilities of CP and Swirling Motion Cores

Flow features over a full heart cycle in a cerebral aneurysm:

- Visualized by nested semi-transparent isosurfaces.
- Streamlines of the mean vector field provide context.

Critical point probabilities with Poincaré index > 0 (blue)
Probabilities for swirling motion cores.
Research Questions in Uncertainty Vis

Uncertainty representations

- Intervals ➔ interval computing
- Probabilities, PDFs ➔ probability theory, statistics
- Fuzzy sets ➔ soft computing
- Dempster-Shafer model ➔ evidence theory
- Possibility model ➔ possibility theory

We Need to Understand …
We Need to Understand …

Reasoning under uncertainty + decision support

- Formal reasoning ➔ statistical inference
- Formal reasoning ➔ uncertainty in AI
- Defuzzification, decision taking ➔ risk & decision theory

To be Developed in Visualization

- UQ in the visualization pipeline
- Fuzzy analogues of crisp features, UQ for features
- Visual mapping of uncertain / fuzzy data
- Evaluation of uncertainty representations, perceptual / cognitive efficiency
- Visual support for data processing techniques:
  - data aggregation, ensemble analysis, …
- Visual support for de-fuzzification
- Visual support in decision making
Conclusion

- Uncertain iso-surfaces, critical points and vortex cores
  - reveals information not visible before
- We (still) rely on assumption of normal distribution
  - arbitrary number of realizations possible
  - more details than with limited number of realizations
- Most important research questions
  - visual mapping
  - non-Gaussian random fields
- Future of Uncertainty Vis

Thank you very much for your attention!

www.zib.de/visual

Slides: http://bit.ly/QmQlqy
http://www.cg.tuwien.ac.at/research/publications/2012/VisWeek-Tutorial-2012-Uncertainty