

# VOTS: VOLUME doTS as a Point-Based Representation of Volumetric Data

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## Abstract

*We present Volume dots (Vots), a new primitive for volumetric data modelling, processing, and rendering. Vots are a point-based representation of volumetric data. An individual Vot is specified by the coefficients of a Taylor series expansion, i.e. the function value and higher order derivatives at a specific point. A Vot does not only represent a single sample point, it represents the underlying function within a region. With the Vots representation we have a more intuitive and high-level description of the volume data. This allows direct analytical examination and manipulation of volumetric datasets. Vots enable the representation of the underlying scalar function with specified precision. User-centric importance sampling is also possible, i.e., unimportant volume parts are still present but represented with just very few Vots. As proof of concept, we show Maximum Intensity Projection based on Vots.*

Categories and Subject Descriptors (according to ACM CCS): I.3.6 [Computer Graphics]: Graphics Data Structures and Data Types

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## 1. Introduction

Volumetric data processing is commonly sample-based. Volumetric models consist of a huge number of samples. Each sample contains only information about the data at one specific position. In order to reconstruct a continuous function, neighborhood connectivity is necessary. Volumetric data is often given on a rectilinear grid. The main advantage hereby is, that the positions of the data samples are stored implicitly and that it allows efficient spatial addressing of the data. The rigid shape of such grids becomes more and more a limitation factor as the data sets are constantly increasing in size due to more advanced acquisition devices. Furthermore, the fraction of non relevant volumetric regions in relation to areas of interest is steadily increasing. Non relevant volumetric regions, such as empty space, should not be represented explicitly. Moreover, this explicit representation needs costly storage resources and introduces additional complexity in processing algorithms. For example, extensive research in volume rendering is devoted to efficiently skipping empty

space. Furthermore the discrete nature of grid representation makes it difficult to apply analytic methods to analyze the data. The maximum along a ray, for example, is often estimated by sampling techniques. A more intuitive solution would be to analytically compute the maximum directly. This would be more exact and more efficient.

In this paper we address these issues and propose a new primitive for volumetric data modelling, processing, and rendering: Volume dots (Vots). Vots are a functional representation of sample-based volumetric data. A Vot comprises the coefficients of a Taylor series expansion, which describes the underlying data of a given region. This approach converts a discrete representation into an implicit representation, and therefore allows to exploit the advantages of analytically processing the data. Vots are a more intuitive and high-level description of the data. They enable the application of focus and context strategies to visualize data and allow to represent regions with different levels of detail. Vots also allow to leverage resources where they are needed, because they can be placed at any position.

The Vot representation opens new ways of data processing, it does not intend to replace the conventional represen-

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tations. Vots are not well suited for very complex datasets with a high level of variation among the samples, in which every sample is of equal importance. However, Vots work well for volumetric data in which only parts of the volume are of great importance. This paper is structured as follows: Section 2 surveys related work. Section 3 presents the general Vots data structure. Section 4 describes the Vot generation. Section 5 shows Maximum Intensity Projection based on Vots as an example application. In Section 6 we conclude our work. Finally, in Section 7 we present ideas for future work.

## 2. Related Work

Recently more and more research is focused on point-based primitives for representation, modeling, processing, and rendering. The main reason for this is the increasing amount of data due to more advanced acquisition devices. Common representations reach their limits in sense of performance and usability, therefore new ways of data representations have to be exploited. Our research is also focused on such a point-based representation and was mainly inspired by the following work:

Levoy [LW] first proposed points as a rendering primitive in the mid-eighties. Following this idea, Pfister et al. [PZvBG00] introduced Surfels as a powerful paradigm to efficiently render complex geometric objects at interactive frame rates. Unlike classical surface discretizations, i.e., triangles or quadrilateral meshes, surfels are point primitives without explicit connectivity. Welsh et al. [WM03] present an algorithm that uses wavelets to convert regular sampled point data to an irregular point hierarchy without reducing the precision of the data. Hopf et al. [HE03] propose a hierarchical splatting algorithm to visualize very large scattered point data at interactive frame-rates. Expensive re-sampling of the data is hereby avoided. Qu et al. [QKSK03] propose a rendering primitive called O-Buffers. It is a flexible structure that stores the positions of arbitrarily distributed samples relative to a regular grid. Rössl et al. [RZNS03] present a new approach to reconstruct non-discrete models from gridded volume samples. As a model, they use quadratic, trivariate super splines on a uniform tetrahedral partition. Csebfalvi et al. [CSK03] propose a volume-rendering technique based on Monte Carlo integration. A point cloud of random samples is generated using a normalized continuous reconstruction of the volume as a probability density function and projected onto a image plane. Lu et al. [LME\*02] present a framework for an interactive direct volume illustration system that simulates traditional stipple drawing. Xie et al. [XWH\*03] address the problem of surface reconstruction of highly noisy point clouds. They fit at each sample point a quadric field which are then blended together to produce a pseudo-signed distance field. Turk et al. [Tur92] propose an automatic method of creating surface models at several

levels of detail from an original polygonal description of a given object.

We extend and integrate these ideas and present a new primitive for volumetric data modelling, processing, and rendering, comprising an efficient compact representation of the underlying volumetric data.

## 3. Vots Data-Structure

Each volumetric dataset can be seen as a volumetric scalar function  $f : \mathcal{U} \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ , where  $\mathcal{U}$  is the domain of  $f$ . In general, it is practically impossible to directly specify this function for a given volumetric dataset. With Vots we propose a piece-wise representation  $\tilde{f}_i : V_i \subseteq \mathcal{U} \rightarrow \mathbb{R}$  of the volumetric scalar function  $f$ .  $V_i$  with  $\bigcup_{i=1..N} V_i \subseteq \mathcal{U}$  partitions the underlying space  $\mathcal{U}$ . Unimportant areas of  $\mathcal{U}$ , e.g., background is omitted. The scalar function over set  $V_i$  is represented by an individual Vot. Analogous to a Taylor series expansion all the relevant information for local function reconstruction is concentrated at a specific point  $P_i$  within a Vot  $V_i$ . Relevant information includes the function value and higher order derivatives, such as gradient and hessian matrix, at a specific point  $P_i$ . Vots are thus a set of points  $\{P_1, P_2, \dots, P_N\}$  in  $\mathbb{R}^3$ . Each point  $P_i = (P_i^x, P_i^y, P_i^z)$  defines a Taylor expansion point and locally represents the scalar volume function via the Taylor series expansion. In general, the Taylor expansion series is defined as:

$$f(P + \Delta P) = \sum_{|\alpha| < n} \frac{1}{\alpha!} \partial^\alpha f(P) \Delta P^\alpha + R \quad (1)$$

with the remainder term:

$$R = \sum_{|\alpha|=n+1} \frac{1}{\alpha!} \partial^\alpha f(P + \theta \Delta P) \Delta P^\alpha \quad (2)$$

for  $\theta \in [0, 1]$ .

For Vots, only terms of the Taylor series expansion up to a specific degree  $N$  are taken into account. We approximate the Taylor series expansion by:

$$f(P + \Delta P) \approx \tilde{f}(P + \Delta P) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial^\alpha \tilde{f}(P) \Delta P^\alpha \quad (3)$$

Due to increasing storage demands and computational complexity with higher degrees, a degree of two or three is a good choice from a practical point of view. The derivatives

up to degree three are given as:

$$\begin{aligned}\partial^1 \tilde{f}(P) &= \nabla f(P) = \begin{pmatrix} \tilde{f}_x \\ \tilde{f}_y \\ \tilde{f}_z \end{pmatrix} \\ \partial^2 \tilde{f}(P) &= H(P) = \begin{pmatrix} \tilde{f}_{xx} & \tilde{f}_{xy} & \tilde{f}_{xz} \\ \tilde{f}_{yx} & \tilde{f}_{yy} & \tilde{f}_{yz} \\ \tilde{f}_{zx} & \tilde{f}_{zy} & \tilde{f}_{zz} \end{pmatrix} \\ \partial^3 \tilde{f}(P) &= T(P) = \begin{pmatrix} \tilde{f}_{xxx} & \tilde{f}_{xyx} & \tilde{f}_{xzx} \\ \tilde{f}_{yxx} & \tilde{f}_{yyx} & \tilde{f}_{yzx} \\ \tilde{f}_{zxx} & \tilde{f}_{zyx} & \tilde{f}_{zzx} \\ \tilde{f}_{xxy} & \tilde{f}_{xyy} & \tilde{f}_{xzy} \\ \tilde{f}_{yyx} & \tilde{f}_{yyy} & \tilde{f}_{yzy} \\ \tilde{f}_{zxy} & \tilde{f}_{zyy} & \tilde{f}_{zzx} \\ \tilde{f}_{xxz} & \tilde{f}_{xyz} & \tilde{f}_{xzz} \\ \tilde{f}_{yxz} & \tilde{f}_{yyz} & \tilde{f}_{yzz} \\ \tilde{f}_{zxz} & \tilde{f}_{zyz} & \tilde{f}_{zzz} \end{pmatrix},\end{aligned}$$

Hereby  $\nabla$  denotes the gradient,  $H$  denotes the Hessian matrix, and  $T$  the third derivative. The Hessian matrix, as well as the third derivative are symmetric. This property can be exploited for storage optimization.

Since each Vot  $\mathcal{V}_i$  represents a certain neighborhood of the volumetric scalar function  $f$ , we also define a validity area  $V_i$ . This area  $V_i$  can be of arbitrary shape. From a practical point of view convex shapes such as spheres, ellipsoids, boxes, k-dops, etc. are advantageous. Each of these validity areas  $V$  is defined in such a way that for a given error  $\epsilon$  (which might be zero) it is always:

$$\forall (P_i + \Delta P_i) \subseteq V_i : |f(P_i + \Delta P_i) - \tilde{f}(P_i + \Delta P_i)| < \epsilon.$$

A basic Vot  $\mathcal{V}_i$  consists of:

- Position  $P_i$ .
- $(\partial^\alpha \tilde{f}(P_i))_{\alpha \in [0, N]}$
- Validity area  $V_i$  for a given  $\epsilon$ .

Vot properties can be extended to include attributes such as as time-step, importance, etc. Vots represent the underlying volume data with data centric importance. Important data areas (i.e., large function variation) are represented with many small Vots (small validity areas  $V_i$ ). Homogenous areas are represented with just a few large Vots. In addition the user may locally vary the approximation error  $\epsilon$ , allowing a user-centric importance sampling. Focus areas are represented with  $\epsilon = 0$ . The context areas might have large errors, which could, for example, increase with distance to the focus area. The Vots data structure concentrates the information where data needs it and the user wants it.

#### 4. Vot Generation

For illustration purposes, before presenting a general method for Vot construction we first show how to directly obtain a Vot representation for a 2x2x2 cell of a rectilinear grid.

As reference reconstruction we assume trilinear interpolation within the cell.

##### 4.1. Vot Generation For A Cell

A cell is given as eight pairs:

$$(P_{ijk}, f_{P_{ijk}})_{i,j,k \in \{0,1\}}$$

Hereby  $P_{ijk}$  denotes a position at one of the corners of the cell, and  $f_{P_{ijk}}$  the corresponding function value. Furthermore we assume a trilinear reconstruction filter. The expansion point needed for the Taylor expansion is defined as center of the cell by:

$$P = \frac{1}{8} \sum P_{ijk}$$

The terms  $\partial^0 \tilde{f}(P)$ ,  $\partial^1 \tilde{f}(P)$ ,  $\partial^2 \tilde{f}(P)$ , and  $\partial^3 \tilde{f}(P)$  for the Taylor series expansion up to degree three can directly be specified by:

$$\begin{aligned}\partial^0 \tilde{f}(P) &= \frac{1}{8} \sum f_{P_{ijk}} \\ \partial^1 \tilde{f}(P) &= \frac{1}{4} \begin{pmatrix} \sum_{i,j} f_{P_{ijk}} - \sum_{i,j} f_{P_{i0j}} \\ \sum_{i,k} f_{P_{i1k}} - \sum_{i,k} f_{P_{i0k}} \\ \sum_{i,j} f_{P_{ij1}} - \sum_{i,j} f_{P_{ij0}} \end{pmatrix} \\ \partial^2 \tilde{f}(P) &= \frac{1}{2} \begin{pmatrix} 0 & \tilde{f}_{xy} & \tilde{f}_{xz} \\ \tilde{f}_{xy} & 0 & \tilde{f}_{yz} \\ \tilde{f}_{xz} & \tilde{f}_{yz} & 0 \end{pmatrix} \\ \partial^3 \tilde{f}(P) &= \tilde{f}_{xyz}\end{aligned}\tag{4}$$

where

$$\begin{aligned}\tilde{f}_{xy} &= \sum_{ijk \in \{000,011,100,111\}} f_{P_{ijk}} - \sum_{ijk \in \{000,001,110,111\}} f_{P_{ijk}} \\ \tilde{f}_{xz} &= \sum_{ijk \in \{000,010,101,111\}} f_{P_{ijk}} - \sum_{ijk \in \{001,011,100,110\}} f_{P_{ijk}} \\ \tilde{f}_{yz} &= \sum_{ijk \in \{000,011,100,111\}} f_{P_{ijk}} - \sum_{ijk \in \{001,010,101,110\}} f_{P_{ijk}}\end{aligned}$$

and

$$\tilde{f}_{xyz} = \sum_{ijk \in \{001,010,100,111\}} f_{P_{ijk}} - \sum_{ijk \in \{000,011,101,110\}} f_{P_{ijk}}$$

Every data value within the cell can be reconstructed by evaluating Equation 3. The Vots representation of a cell requires altogether eight values.  $\partial^0 \tilde{f}(P) \rightsquigarrow 1$ ,  $\partial^1 \tilde{f}(P) \rightsquigarrow 3$ ,  $\partial^2 \tilde{f}(P) \rightsquigarrow 3$ , and  $\partial^3 \tilde{f}(P) \rightsquigarrow 1$ . All the remaining values are either redundant due the symmetry of the hessian matrix or are zero. In terms of storage requirement a Vot representation is equal to a cell representation. However, in a regular grid, grid points are reused for (eight) neighboring cells. A straightforward conversion of a regular grid into a Vot structure would therefore increase the storage requirements by a factor of eight. A closer look reveals that only every other cell (in each of the three spatial directions) must be represented by a Vot. The function in cells which do not contain a Vot can be exactly reconstructed from the neighboring Vots. Such an approach would not change the

storage requirements. With the Vots representation we have a more intuitive representation of the underlying function.

#### 4.2. General Vot Generation

One of the major reasons for the introduction of Vots is to be able to leverage resources where they are needed. Homogeneous or non important regions should be represented just by few resources. On the other hand, inhomogeneous and important regions should be represented by an adequate amount of resources. The amount is defined by the desired accuracy. Vots provide this feature, as they can be placed at any arbitrary position.

In the following we present an approach to generate a Vot for a given set of  $m$  scattered data points  $Q_j \in \mathbb{R}^3$  with data values  $f_{Q_j}$ . We assume that a Vot uses the Taylor expansion series up to degree  $N = 3$ , as shown in Equation 3. This can be extended to arbitrary degrees straightforwardly. To generate the Vots, we use an approach which is similar to the linear regression approach for normal vector estimation used in [NCKG00]. To be able to apply this approach we define the mean square error of the fitting process as:

$$E(\dots) = \sum_{j=1}^m (\tilde{f}(Q_j) - f_{Q_j})^2 \quad (5)$$

Hereby  $f_{Q_j}$  denotes the function values given by the  $m$  scattered points,  $\tilde{f}(Q_j)$  denotes the function value reconstructed by the approximated Taylor function series  $\tilde{f}$  at  $Q_j$ , and  $E$  is the sum of the squared differences between the original values and the reconstructed values. As Taylor series expansion point we choose the center of gravity:

$$P = \frac{1}{m} \sum Q_j$$

The unknown variables of  $E$  are:

$$\tilde{f}, \tilde{f}_i, \tilde{f}_{ij}, \tilde{f}_{ijk}$$

where  $i, j, k \in \{x, y, z\}$  and  $\tilde{f}_i, \tilde{f}_{ij}, \tilde{f}_{ijk}$  denote the partial derivatives  $\partial_i \tilde{f}, \partial_i \partial_j \tilde{f}, \partial_i \partial_j \partial_k \tilde{f}$ . The unknowns can be reduced due to symmetry of the hessian matrix and the third derivative to 20 unknowns. Furthermore we define:

$$\Delta Q_j = Q_j - P = (Q_j^x - P^x, Q_j^y - P^y, Q_j^z - P^z)$$

The minimum of the error function  $E$  is determined by taking the partial derivatives with respect to the unknowns and setting these partial derivatives to zero. To achieve this,  $\tilde{f}(Q_j)$  is substituted according to Equation (3) with

$$\sum_{i=0,\dots,3} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta Q_j^i$$

The derivatives are given as:

$$\begin{aligned} \frac{\partial E}{\partial \tilde{f}} &= 2 \sum_{j=1}^m \left( \sum_{i=0,\dots,3} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta Q_j^i - f_{Q_j} \right) \\ \frac{\partial E}{\partial \tilde{f}_x} &= 2 \sum_{j=1}^m \left( \sum_{i=0,\dots,3} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta Q_j^i - f_{Q_j} \right) \Delta Q_j^x \\ &\vdots \\ \frac{\partial E}{\partial \tilde{f}_{xy}} &= 2 \sum_{j=1}^m \left( \sum_{i=0,\dots,3} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta Q_j^i - f_{Q_j} \right) \Delta Q_j^x \Delta Q_j^y \\ &\vdots \\ \frac{\partial E}{\partial \tilde{f}_{xxy}} &= 2 \sum_{j=1}^m \left( \sum_{i=0,\dots,3} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta Q_j^i - f_{Q_j} \right) (\Delta Q_j^x)^2 \Delta Q_j^y \\ &\vdots \end{aligned} \quad (6)$$

Setting the derivatives to zero, leads to the following system of linear equations

$$M \begin{pmatrix} \tilde{f} \\ \tilde{f}_x \\ \vdots \\ \tilde{f}_{xy} \\ \vdots \\ \tilde{f}_{xxy} \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m f_{Q_j} \\ \sum_{j=1}^m f_{Q_j} \Delta Q_j^x \\ \vdots \\ \sum_{j=1}^m f_{Q_j} \Delta Q_j^x \Delta Q_j^y \\ \vdots \\ \sum_{j=1}^m f_{Q_j} \Delta Q_j^x \Delta Q_j^x \Delta Q_j^y \\ \vdots \end{pmatrix} \quad (7)$$

where  $M$  is the  $20 \times 20$  matrix resulting from the following sum of vector direct products:

$$M = \sum_{j=1}^m \begin{pmatrix} 1 \\ \Delta Q_j^x \\ \Delta Q_j^y \\ \Delta Q_j^z \\ (\Delta Q_j^x \Delta Q_j^x)/2 \\ \Delta Q_j^x \Delta Q_j^y \\ \Delta Q_j^x \Delta Q_j^z \\ (\Delta Q_j^y \Delta Q_j^y)/2 \\ \Delta Q_j^y \Delta Q_j^z \\ (\Delta Q_j^z \Delta Q_j^z)/2 \\ (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^x)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^y \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^y \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^z \Delta Q_j^z)/6 \\ \Delta Q_j^x \Delta Q_j^y \Delta Q_j^z \\ (\Delta Q_j^y \Delta Q_j^y \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^y \Delta Q_j^y \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^y \Delta Q_j^z \Delta Q_j^z)/6 \\ (\Delta Q_j^z \Delta Q_j^z \Delta Q_j^z)/6 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \Delta Q_j^x \\ \Delta Q_j^y \\ \Delta Q_j^z \\ (\Delta Q_j^x \Delta Q_j^x)/2 \\ \Delta Q_j^x \Delta Q_j^y \\ \Delta Q_j^x \Delta Q_j^z \\ (\Delta Q_j^y \Delta Q_j^y)/2 \\ \Delta Q_j^y \Delta Q_j^z \\ (\Delta Q_j^z \Delta Q_j^z)/2 \\ (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^x)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^x \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^y \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^y \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^x \Delta Q_j^z \Delta Q_j^z)/6 \\ \Delta Q_j^x \Delta Q_j^y \Delta Q_j^z \\ (\Delta Q_j^y \Delta Q_j^y \Delta Q_j^y)/6 \\ 3 \cdot (\Delta Q_j^y \Delta Q_j^y \Delta Q_j^z)/6 \\ 3 \cdot (\Delta Q_j^y \Delta Q_j^z \Delta Q_j^z)/6 \\ (\Delta Q_j^z \Delta Q_j^z \Delta Q_j^z)/6 \end{pmatrix}^T \quad (8)$$

The inversion of matrix  $M$  produces the solution for the

unknown variables. The error is calculated by

$$\varepsilon = \max_j |\tilde{f}(Q_j) - f_{Q_j}|$$

The mechanism described allows the construction of a Vot for a given input set of Points  $Q_j, j = 1, \dots, N$ .

### 4.3. Vot-Space

Vots allow an importance-based representation of volume data. To achieve this they abandon the implicit connectivity information of regular grids. The most basic question the data structure has to answer is: Given an arbitrary point  $P$  find the corresponding Vot  $V_i$  ( $P \in V_i$ ) so that the function value at position  $P$  can be determined. Depending on the shape of the validity area  $V_i$  efficient indexing structures from computational geometry, such as range trees, interval trees, octrees and bounding volume hierarchies can be used to accelerate this search. Application dependent, indexing structures might also help to quickly address, for example, all the Vots whose gradient is within a certain magnitude or direction range. Here it comes in handy that certain data characteristics like gradient or curvature (via the hessian matrix), which are often used in volumetric data processing, are directly available through the Vots data-structure.

A Vot-Space  $(\mathcal{V}_j, \mathcal{I})$  comprises a set of Vots  $\mathcal{V}_j$  and a set of indexing structures  $\mathcal{I}$ .  $\mathcal{I}$  contains at least  $I^*$ , which is an unsorted list of all Vots. It used to address each Vot. For some application this simple indexing structure is sufficient, see Section 5. The kind of indexing structures a Vot-Space is dependent on the specific application.

## 5. Application

To give a proof of concept of our new data structure, we show an application of Vots. We present Maximum Intensity Projection of a Vot-Space. Maximum Intensity Projection [MGK99] is a technique that displays the maximum scalar value seen through each image pixel. By depicting the maximum data value, high intensity structures contained in the data are captured. A straight-forward method for calculating Maximum Intensity Projection is to perform ray casting and search for the maximum sample value along each ray. We use this visualization method to illustrate the advantages of Vots. Instead of using sampling, we determine the maximum of a viewing ray within the validity area of a Vot analytically.

### 5.1. Vots Generation

As input we assume a rectilinear grid. It is given as a set of pairs  $G = \{(P_j, f_{P_j}), j = \{1, \dots, N\}\}$  where  $P_j = (P_j^x, P_j^y, P_j^z)$  defines a position within the grid and  $f_{P_j}$  the corresponding function value  $f(P_j)$ . The task is to find a small number of Vots  $V_j$  which covers completely the underlying volumetric data. Reconstruction should be bound by an error  $\varepsilon$ , i.e.

$$\forall (P_j + \Delta P_j) \subseteq V_j : |f(P_j + \Delta P_j) - \tilde{f}(P_j + \Delta P_j)| < \varepsilon. \quad (9)$$

for a given error bound  $\varepsilon$ .

We use a growing approach. As growing criteria we define the function  $\mathcal{F}(T)$  as follows:

$$\mathcal{F}(T) = \max_j |\tilde{f}(Q_j) - f_{Q_j}|, Q_j \in T$$

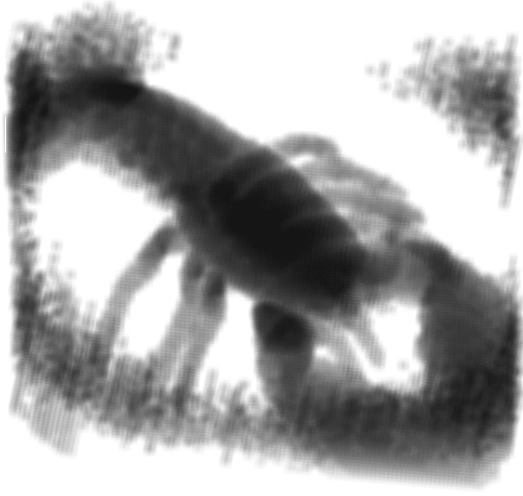
where  $T \subseteq G$ .  $\mathcal{F}$  implements Equation 7, it solves the linear equation system and returns the maximal error. Furthermore, for simplicity we assume box-shaped validity areas of the Vots. As possible start positions of Vots we choose the positions given by  $G$ . The validity area of each Vot is initially cell-sized, covering eight adjacent grid positions. The validity of this assumption, was shown in Section 4.1. At this point, we iteratively start the growing process of each Vot by increasing one of its validity area dimensions either in the positive or negative direction. In every step we test the error computed by  $\mathcal{F}$ . The current input set  $T$  is defined by all grid positions which lie within the increased validity area. According to the outcome of  $\mathcal{F}$  and the given *epsilon*-bound we either keep the increased validity area as new area or we keep the old validity area. The growing process stops once no Vot can increase its area without breaking the condition given by Equation 9.

The result of this process is a set of the largest Vots for every position. The next step is to find a minimal subset of the set of Vots which completely cover the underlying volumetric data. To achieve this we assign to each Vot  $V_j$  a cover weight  $\mathcal{W}_j$ . The weights initially correspond in size to the validity areas  $V_j$ . The Vots are sorted according to their weights  $\mathcal{W}_j$  in decreasing order. We take the Vot with the largest  $\mathcal{W}$  and assign it to the subset of the minimal Vots. We determine the number of grid points this Vot would cover in the grid. Only those grid positions are counted, which are not covered by other Vots in the current optimal subset. We adapt all the weights  $\mathcal{W}_j$  of all the remaining Vots, according to the number of grid positions they could cover, which are not already covered by other Vots out of the current optimal subset. Then the Vots are re-sorted and the process starts all over. Once the complete grid is covered the process stops and we found a minimal subset of Vots which defines our Vot-Space with a given indexing structure  $I^*$ .

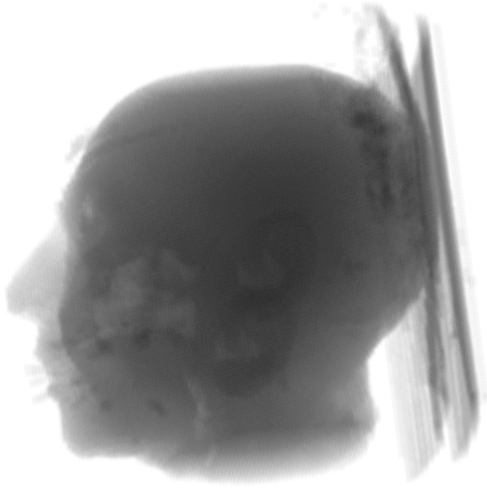
Figure 1 and Figure 2 show the Vot distribution of typical datasets. Dark areas correspond to high Vot density and bright areas to a low Vot density.

### 5.2. Maximum Intensity Projection Of A Vot-Space

The first question which arises is: Given a Vot  $V$  and a viewing direction  $R$ , how do we compute the maximum along this ray. To answer this, we first look again at the Taylor series



**Figure 1:** Vot density distribution of lobster dataset: 113.111 Vots generated from 522.784 input samples



**Figure 2:** Vot density distribution of UNC head dataset: 562.825 Vots generated from 1.835.008 input samples

expansion of one Vot. It is given as:

$$\tilde{f}(\Delta P) = \sum_{i=0, \dots, N} \frac{1}{i!} \partial^i \tilde{f}(P) \Delta P^i$$

where  $P$  is the corresponding Taylor expansion point. For simplicity reasons we choose  $N = 2$ , from this follows that

the Taylor series is given as:

$$\begin{aligned} \tilde{f}(P + \Delta P) &= \tilde{f}(P) + \partial^1 \tilde{f}(P) \Delta P + \frac{1}{2} \partial^2 \tilde{f}(P) \Delta P^2 \\ &= \tilde{f}(P) + \nabla(\tilde{f}(P)) \Delta P + \frac{1}{2} \Delta P H(P) \Delta P \end{aligned} \quad (10)$$

A ray  $R$  is given as  $S + tD$  and  $\Delta P = X - P$ . Setting the ray equation into  $X - P$ , we obtain  $S + tD - P$ . We substitute  $S - P$  with  $\tilde{S}$  and set  $\tilde{S} + tD$  into Equation 10. To determine the extreme we take the first derivative and obtain:

$$\partial^1 \tilde{f}(\tilde{S} + tD) = a_0 + a_1 t$$

with

$$\begin{aligned} a_0 &= \tilde{f}_x R_x + \tilde{f}_y R_y + \tilde{f}_z R_z + R_x \tilde{S}_x (\tilde{f}_{xx} + \tilde{f}_{xy} + \tilde{f}_{xz}) + \\ &\quad R_y \tilde{S}_y (\tilde{f}_{xy} + \tilde{f}_{yy} + \tilde{f}_{zy}) + R_z \tilde{S}_z (\tilde{f}_{zy} + \tilde{f}_{xz} + \tilde{f}_{zz}) \\ a_1 &= R_x^2 (\tilde{f}_{xx} + \tilde{f}_{xy} + \tilde{f}_{xz}) + R_y^2 (\tilde{f}_{xy} + \tilde{f}_{yy} + \tilde{f}_{zy}) + \\ &\quad R_z^2 (\tilde{f}_{zy} + \tilde{f}_{xz} + \tilde{f}_{zz}) \end{aligned}$$

The position of the extreme is then determined by setting the first derivative to zero. The resulting equation is solved with respect to  $t$ :

$$t = \frac{-a_0}{a_1}$$

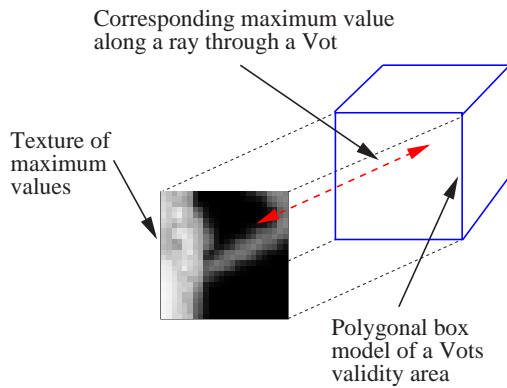
To determine whether the extreme is a maximum or minimum, we set  $\tilde{S} + tD$  into Equation 10, take the second derivative and obtain:

$$\begin{aligned} \partial^2 \tilde{f}(\tilde{S} + tD) &= R_x^2 (\tilde{f}_{xx} + \tilde{f}_{xy} + \tilde{f}_{xz}) + \\ &\quad R_y^2 (\tilde{f}_{xy} + \tilde{f}_{yy} + \tilde{f}_{zy}) + \\ &\quad R_z^2 (\tilde{f}_{zy} + \tilde{f}_{xz} + \tilde{f}_{zz}) \end{aligned}$$

The sign of the second derivative determines if the extreme is a maximum or a minimum. Knowing these two derivatives it is straightforward to determine the maximum along a ray within the validity area of a Vot. Two cases are distinguished:

1. A maximum within the validity area is found.
2. No maximum within the validity is found, the maximum along the ray occurs at one of the intersection points with the validity area box.

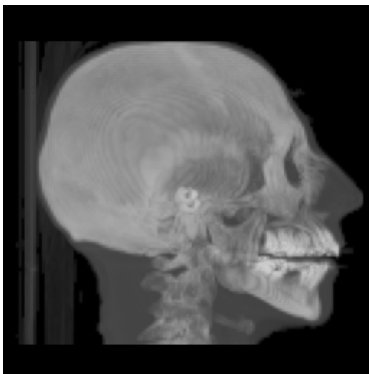
With this method, the maximum scalar value of an arbitrary ray passing through a Vot can be determined. The algorithm now works as follows: The unsorted list of Vots is traversed. For each Vot, its validity region is represented by a polygonal model. For every ray that intersects the Vot, the intersection point  $S$  is calculated and the maximum along the ray is computed and stored in an image. There is one image for each visible face of the model. In our case, the validity area is always box-shaped, therefore at most three faces are visible. Texture mapping is used to transform the images according to the validity area geometry. The images are textured onto the corresponding faces of the model, as illustrated in Figure 3. We utilize the graphics hardware's capability to perform maximum blending. With our prototype application we rendered two example datasets shown in Figure 4 and Figure 5.



**Figure 3:** Maximum Intensity Projection of one Vot.



**Figure 4:** Maximum Intensity Projection of 113.111 Vots (Lobster dataset)



**Figure 5:** Maximum Intensity Projection of 562.825 Vots (UNC head dataset)

## 6. Conclusion

We proposed a novel primitive for volumetric data modelling, processing, and rendering. As we moved the data representation from a discrete to an implicit representation,

a new paradigm was presented. The new function oriented paradigm is a more intuitive and constructive representation of the data. The volumetric data is divided into regions to achieve a more compact representation. Some Vots represent larger regions than others, but all Vots represent the data in the same way. The size of a Vot can be adjusted by modifying the allowed error bound  $\epsilon$ . This allows user-centric importance sampling. Unimportant regions are represented by just a few Vots, while important regions are represented with many Vots. One Vot contains all the information about the volumetric data within a region, thus no explicit connectivity between Vots is necessary for reconstruction. Furthermore, the Vots representation allows to process the data analytically as shown with Maximum Intensity Projection. In general, Vots open a wide range of new data examination approaches.

## 7. Future Work

In the future we will continue to explore the new possibilities and approaches that the new paradigm of Vots introduces. One of the first challenges is to construct Vots from any other kind of data structures, such as point clouds, unstructured grids, curvilinear grids, and others. Different strategies must be developed to obtain an accurate conversion. Vots provide a starting point for new types of volume processing, new ways of rendering, even exploring new types of visualization based on the condensed information that a Vot contains. It would also be interesting to explore the possibility of mapping Vots onto graphics hardware and analyzing its capabilities from that point of view.

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