Pseudo-Cartograms †

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ABSTRACT. Pseudo-cartograms provide a convenient approximation to pycnomirastic map projections and are easily computed from one dimensional integrals of marginal density distributions Examples are given using world population.

KEY WORDS map projections, uniformizing maps density smoothing

It is well known that the geographically useful class of map projections known as cartograms are defined by the partial differential equation

$$\partial x/\partial\lambda \; \partial y/\partial\varphi - \partial x/\partial\varphi \; \partial y/\partial\lambda - D(\varphi, \lambda)\, R^2 \cos\varphi = 0, \qquad (1)$$

where, as usual, φ represents latitude, λ longitude, x and y are cartesian coordinates, and D (φ, λ) is the relevant density on the sphere of radius R (Tobler 1963). When the density D (φ, λ) is constant we obtain the usual defining equation for an equal area map projection of the sphere (Lambert 1772). In both cases the single definitional equation does not suffice because we need to obtain the two functions $x = f(\varphi, \lambda)$, $y = g(\varphi, \lambda)$ that define the mapping; this is true for equal area projections and for the mass distributing cartograms. As for any partial differential equation, it is also necessary to provide boundary conditions.

As an important secondary objective, necessary to completely, specify a cartogram, it has been proposed (Tobler. 1973b) that the map be as nearly conformal as possible, and this again minimizes the condition for the 'best' equal area projection, as defined, inter alia, by Mescheryakov (1962,1968). This latter problem, finding the equal area projection of minimum total angular distortion remains unsolved in the general case. The uneven arrangement of phenomena on the sphere, D (φ, λ) in equation (1), makes the pycnomirastic map projection problem even more difficult than the unsolved equal area one. Of course one might specify simpler conditions for the cartogram, similar to the selection of cylindrical, conical, or azimuthal equal area projections, but the near conformality condition has a great deal of intuitive appeal. It makes the cartogram look as nearly as possible like the more conventional map, and minimizes the directional variation in the distortion of distances at any point, and this is important for geographic applications (Bunge 1965). Thus shapes are more nearly preserved locally (Sen 1976).

Mathematica1ly it turns out that the entire problem is somewhat easier if it is inverted and specified as Minimize:

$$\int_{-\pi}^{+\pi} \int_{-\pi/2}^{+\pi/2} \left[\left(\frac{\partial x}{\partial\varphi}^2 + \frac{\partial y}{\partial\varphi}^2 \right) \cos\varphi + \left(\frac{\partial x}{\partial\lambda}^2 + \frac{\partial y}{\partial\lambda}^2 \right) \right] d\lambda \, d\varphi \qquad (3)$$

subject to the pycnomirastic condition (1) and to boundary conditions. Formally this is an optimization problem with a non-linear constraint, in principle solvable by Lagrangian multiplier methods. Based on these ideas an iterative algorithm, described elsewhere (Tobler 1973b, 1976), has been coded for numerical solution (Tobler 1973a). The present note is concerned solely.with the choice of an efficient initia1 configuration from which The iterative computation can proceed.

If the density of the phenomena on the sphere could be written as a separable function:

$$D(\varphi, \lambda) = D(\varphi)\, D(\lambda)$$

then one would expect that the problem would be greatly simplified. In particular, the 'natural' and simplest condition then would be to set

$$\partial x/\partial\varphi = \partial y/\partial\lambda = 0$$

and to use a solution of (1) obtained by direct integration, e.g.,

$$y = R\int_{-\pi/2}^{+\pi/2} D_1(\varphi)\cos\varphi\, d\varphi,\quad x = R\int_{-\pi}^{\lambda} D_2(\lambda)\, d\lambda.$$

In most interesting cases it unfortunately will not be possible to describe the geographic arrangement of phenomena on the spherical earth as a separable function. I have been able to imagine only a few instances in which it is plausible to do so. One case is the population density of a country such as Sweden that, to a good approximation, varies only in a north-south direction. What east-west variation exists is essentially independent of the primary variation. There may be additional instances of such latitudinally related phenomena. A second example is the hypothetical circularly symmetric population density of a single-center city. In such a case the density function is separable in polar coordinates. An example of this is given in Tobler (1963). In these cases a pycnomirastic map projection can be obtained by direct integration.

Fortunately, it is possible to write most functions as a sum of products of separable functions. Specifically, suppose that one has geographic data accumulated in the cells of an m by n array Z. Then it is known (Baglai and Smirnov 1975) that this can be written as the sum of k arrays (k ≤ the lessor of m, n) of order m by n, each of which is in turn the product of an m by one vector U_k multiplied by a one by n vector V_k. i.e., the model is:

$$Z = U_1V_1 + U_2V_2 + \ldots + U_kV_k \qquad (7)$$

The U's can be interpreted as functions of the row index only, the V's as functions of the column index, and we can identify the y coordinate with the row index and the x coordinate with the column index. The column vector U_k is the k^{th} eigenvector of $Z Z^t$, and the row.vector V_k is obtained as

$$V_k = Z^t U_k / U^t U_k$$

where t denotes the transpose. The products U_kV_k can be arranged so that the amount of variance accounted for by the model ranges from maximum to minimum. A computer program for this data decomposition is given in Tobler (1970, pp. 123-131). In words, we can pick a single product (the first one) of separable functions that best approximates our given density arrangement in a particular system of coordinates. In theory we should rotate to principal axes before doing this and choose an appropriate coordinate system (polar, rectangular, elliptical, etc.). The pseudo-cartogram depicts this first term of the eigenvector expansion.

In practice the pseudo-cartogram is obtained as

$$y = R\int D(\varphi, \lambda)\cos\varphi\, d\varphi, \qquad x = R\int D(\varphi, \lambda)\, d\lambda \qquad (9$$

with integration limits as before. It is referred to as a 'pseudo' cartogram because it only provides an approximation to the true pycnomirastic solution. The resulting map (in rectangular coordinates) has the form of a cylindrical projection

$$y = f(\varphi),\ x = g(\lambda)\ (10)$$

and equation (1) is not satisfied. But it provides the best possible, in the least squares sense, approximation to a solution as can be obtained by a product of the form

$$D(\varphi, \lambda) \approx D(\varphi)\, D(\lambda) = \partial f/\partial\varphi\ \partial g/\partial\lambda\ R^2\cos\varphi \qquad (11).$$

Thus, it provides a convenient starting point for the iterative algorithm.

Pseudo-cartograms therefore are approximate cartograms. As such they make interesting maps in their own right and are certainly easy to compute.[2] Several examples are presented here to further illustrate some of the ideas involved in the computation. In the first instance, Figure 1 approximates the population of the world, by five degree quadrilaterals of latitude and longitude, on the square projection. Figure 2 shows these values summed by latitude and longitude, and the cumulants representing equations (9) and (10). A world map made from these values is shown in Figure 3; the detailed graticule for this map is shown in Figure 4. The next figure uses the same data to draw a pseudo-cartogram in polar coordinates centered on the North Pole. Figure 6 represents a version based on a sinusoidal projection with a pole line (Wagner 1962). Below this is shown the latitude and longitude graticule for a world cartogram that started from this initial approximation. The starting configuration is clearly retained in the resulting cartogram.

The example (Figure 8) using the United States illustrates some of the difficulty. Nevada is much too large as a consequence of the marginal distributions. But the error of this approximation is less than would be obtained if the square projection were to be used to represent the population distribution directly. This relates to the efficacy of pseudo-cartograms in the complete cartogram computation. Without this step one would initialize the iterations using latitude and longitude coordinates. This is equivalent to the use of the square projection (plate carrée) in rectangular coordinates. In this initial configuration there is a departure from the desired values of

$$R^2 - R^2 D(\varphi, \lambda) \cos \varphi$$

at each point of the map. If the integral of this value over the entire region of interest is taken to represent a 100% error[3] then the pseudo-cartogram immediately reduces this error to approximately 60% in the case of the US population - the particular reduction depends on the separability of the empirical distribution in the initial coordinates. A forty percent error reduction saves a great deal of computer time in the iterations for the final cartogram. It is worth remarking that this approach via pseudo-cartograms does not minimize the function given in equation (3), and thus that other improvements seem possible.

The description given here for the computation of pseudo-cartograms proceeds most easily when the data are given in the form of values for latitude/longitude quadrilaterals. When the data arrive in the form of irregular spherical polygons - e.g., countries or states - then the same procedures may be used except that the marginal sums and numerical integrations are based on the centroids of these polygons. The computer program (Tobler 1973a) used to produce contiguous pycnomirastic map projections from irregular polygonal regions contains an option to use such pseudo-cartograms for the initial configuration of the iterative calculations. Alternately, pycnophylactic interpolation (Tobler 1979) may be used to smoothly assign the population values to a fine lattice within each of the regions. These lattice data may then be used to compute the map projection.

NOTES

1. The terminology is not standardized. The term "value-by-area map" is used by Raisz (1938). The Soviets call them "varivalent projections," the French "anamorphoses," a mathematician uses "uniformizing maps." "Mass distributing (pycnomirastic) map projections" is proposed for this class, and is consistent with recognized usage.
2. A computer listing is available from the author.
3. In practice it is more convenient to use a root mean square error criterion.

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Figure 1: Approximate world population by five degree quadrilaterals of latitude and longitude. Assembled by the author, circa 1965.

Figure 2: Approximate world population summed by latitude (left) and by longitude (right) and cumulants thereto. Based on the same data as Figure 1.

Figure 3: Pseudo-cartogram based on the values in Figure 2.

Figure 4: Latitude, longitude graticule (five degree intervals) corresponding to the map in Figure 3.

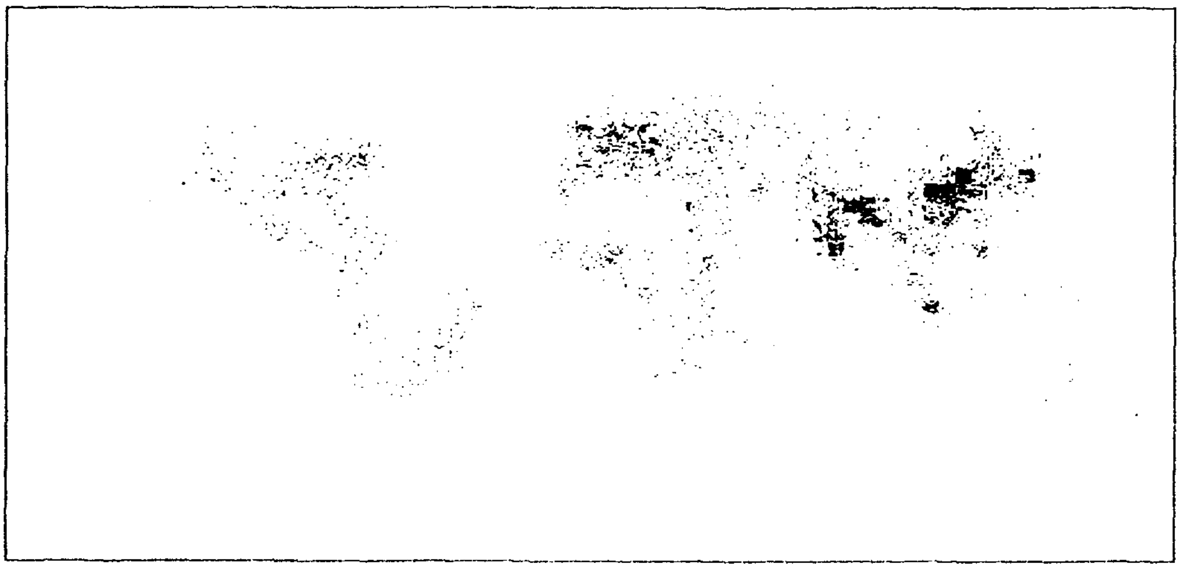
Figure 5: Pseudo-cartogram, in polar form, based on the data in Figure 2.

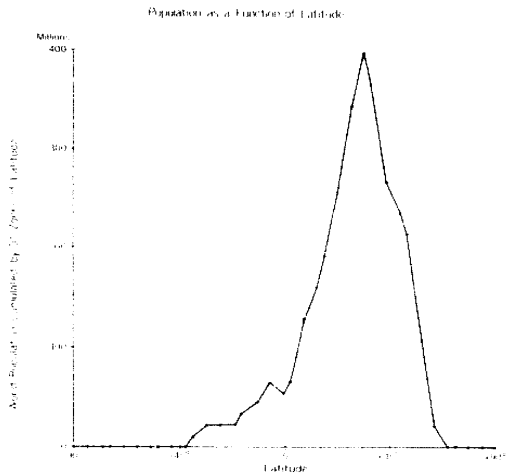
Figure 6: As in Figure 4 but based on a sinusoidal projection with a pole line. Slightly fudged so that empty areas do not disappear completely.

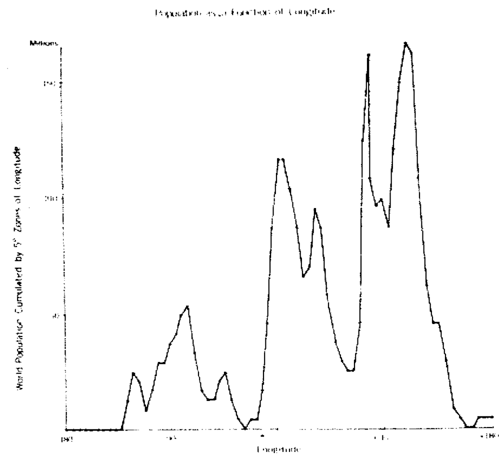
Figure 7: Latitude and longitude for a pycnomirastic map projection obtained from Fig. 6 by iteration to

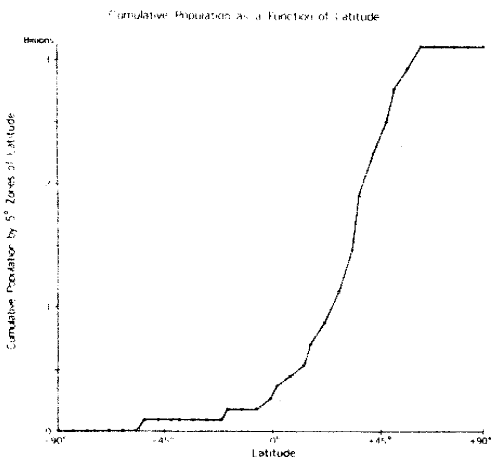
fit the data of Fig. 1. Notice the residual impact of the sinusoidal projection, and the float of the boundaries.

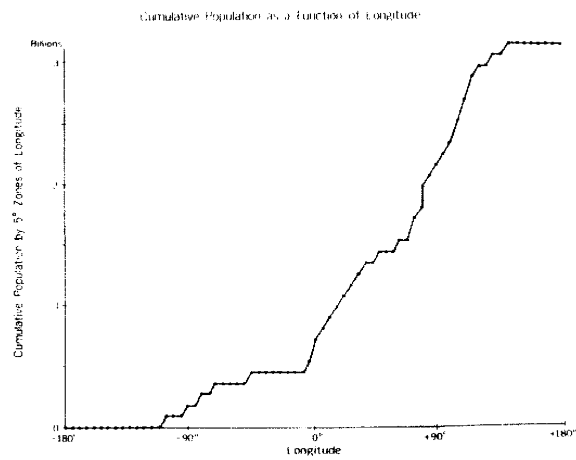
Figure 8: Pseudo-cartogram of the United States obtained from population data given by one degree quadrilaterals.




Population as a Function of Latitude
Millions
400
Latitude


Millions
World Population Cumulated by 5° Zones of Longitude


Cumulative Population as a Function of Latitude
Billions
Cumulative Population by 5° Zones of Latitude
0
-90°
-45°
0°
+45°
+90°
Latitude


Cumulative Population as a Function of Longitude
Billions
Cumulative Population by 5° Zones of Longitude
0
-180°
-90°
0°
+90°
+180°
Longitude