… Sampling …
… Filtering …
… Reconstruction …

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Commission for Scientific Visualization
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Motivation example (2)

Motivation example (3)

Motivation example (4)

Poor picture

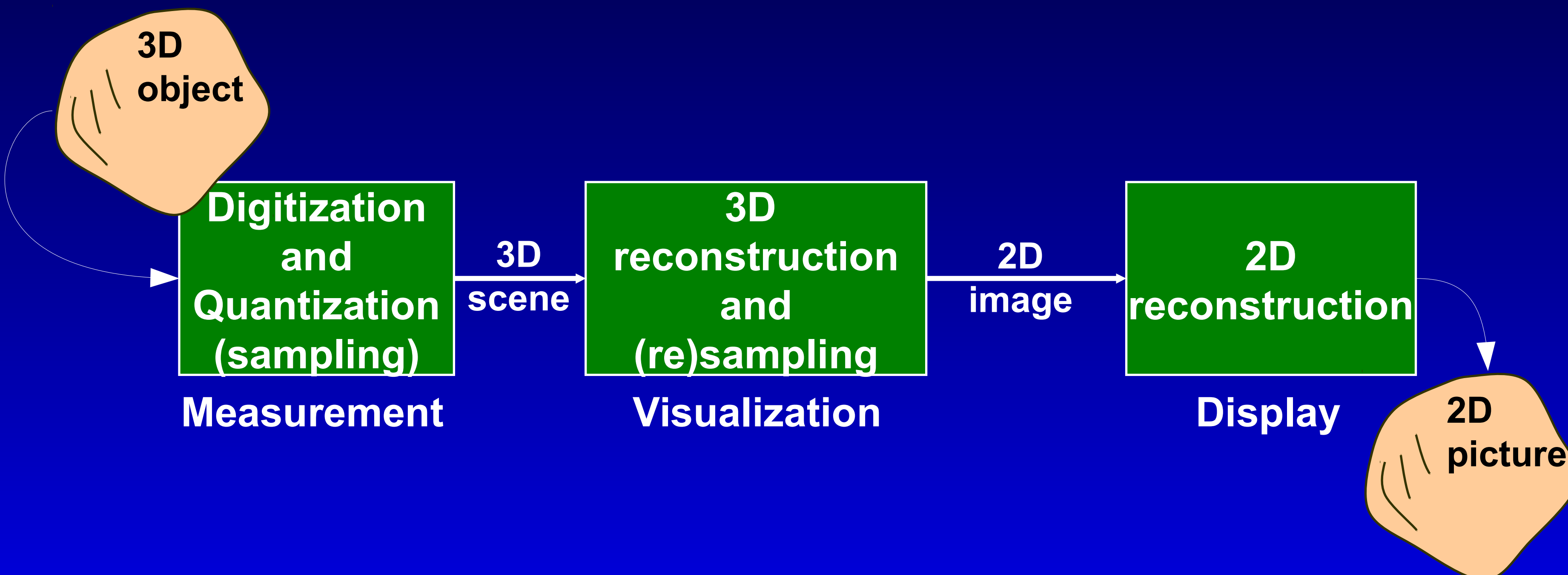
Nice picture

Overview

- Sampling of continuous signals
- Filtering of discrete signals
- Reconstruction of continuous signals
- Sampling aspects of volume rendering
- Voxelization of geometric objects

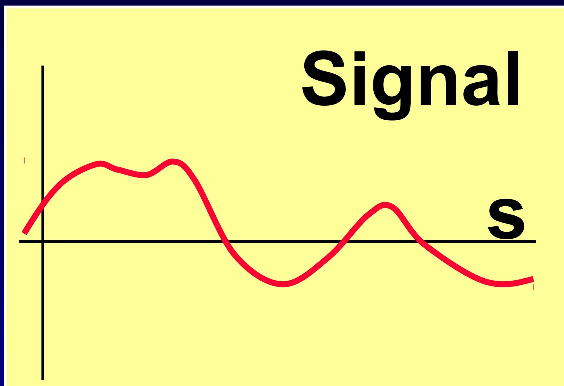
Why Sampling and Reconstruction?

- Real-world signals are continuous



- Computer representations are discrete

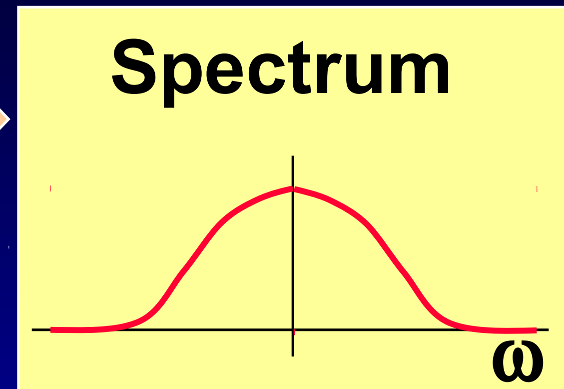
Signal and Spectrum



Spatial / time domain

Fourier transformation

$$f(s) \leftrightarrow F(\omega)$$



Frequency/spectral domain

- FT: decomposition of a signal into a sum of sinusoids, determined by **frequency, amplitude** and **phase**
- *f(x)* - representation in the **spatial domain**
- *F(ω)* - representation in the **frequency domain**

Fourier Transform

$$\boldsymbol{f(s) \leftrightarrow F(\omega)}$$

- **Direct FT:**

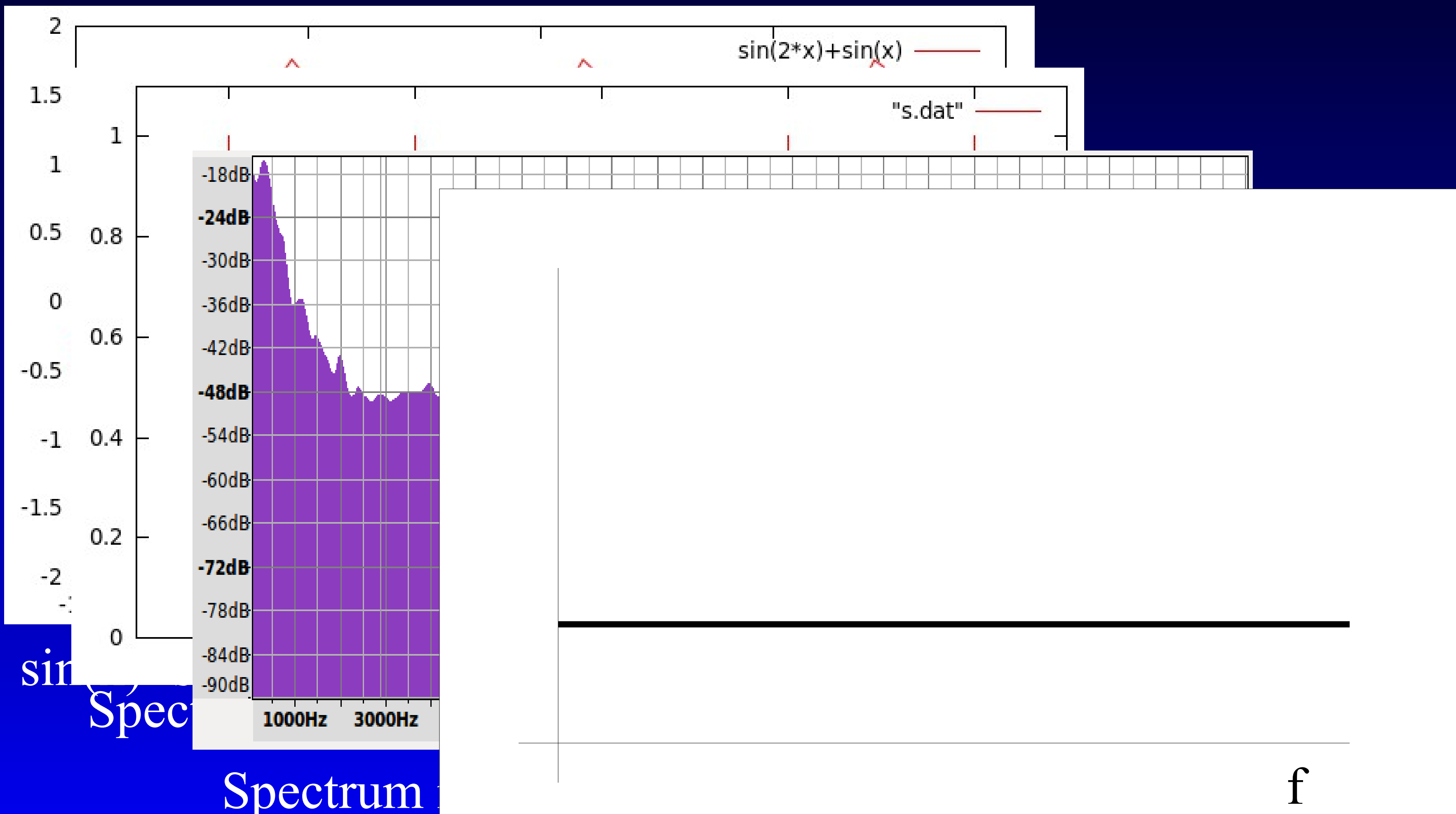
$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-i\omega s} ds$$

- **Inverse FT:**

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega s} d\omega$$

$$\omega = 2\pi f$$

Examples



sin

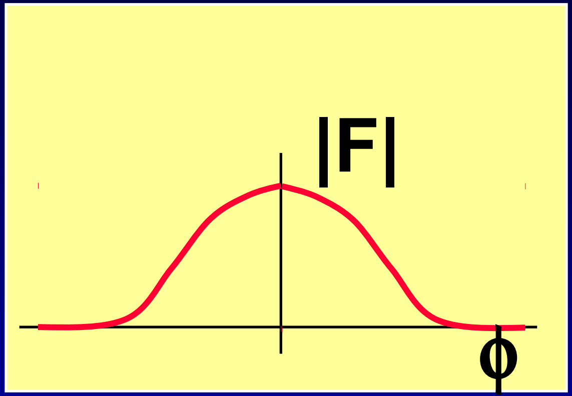
Spec

Spectrum

Spectrum white noise

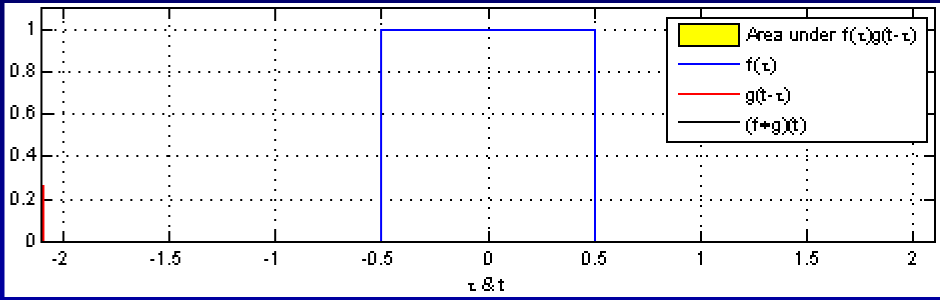
Typically Drawn Spectrum

- Symmetric
  - F is complex, we draw |F|
- Falling towards high frequencies
- Band limited: has maximum frequency ω_{max}



Convolution (1)

- f * h : a weighted sum of *f* with weight function *h* placed at *s*



- Applications:
  - Smoothing, noise removal, edge detection, ...
- Other names: filter (e.g. lowpass)

Convolution (2)

- **Continuous case:**

$$f(s) * h(s) = \int_{-\infty}^{\infty} f(s) \cdot h(s-\tau) d\tau$$

- **Discrete case:**

$$f[n] * h[n] = \sum_{k} f[k] \cdot h[n-k]$$

f: signal, h: convolution kernel

Convolution Theorem

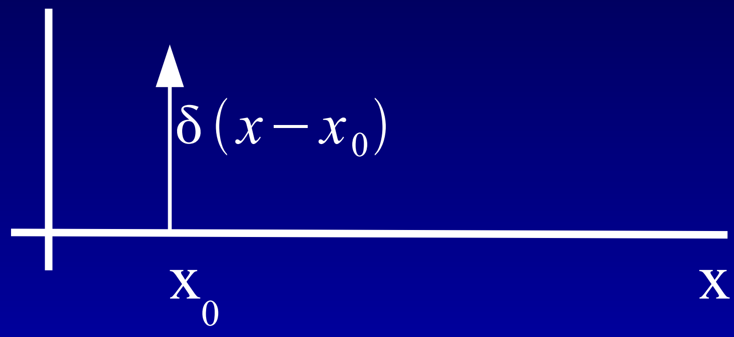
- Convolution in spatial (frequency) domain corresponds to multiplication in frequency (spatial) domain

$$FT\{f(t) * h(t)\} = F(\omega) \cdot H(\omega)$$

$$FT\{f(t) \cdot h(t)\} = F(\omega) * H(\omega)$$

Dirac's δ-function

- Can be used to describe the sampling process
- Properties:



$$\int_{-\infty}^{\infty}\delta(x)=1$$

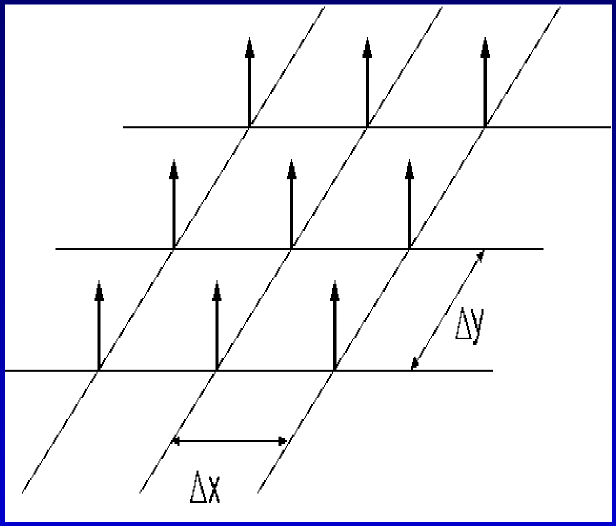
$$f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)$$

$$\int_{-\infty}^{\infty}f(x)\delta(x-x_0)=f(x_0)$$

Sampling

- Input signal: $f_I(x, y)$
- Ideal sampling function:

$$s(x,y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(x - i\Delta_x, y - j\Delta_y)$$



- Sampled signal:

$$f_s(x,y) = f_I(x,y)s(x,y)$$

Sampling in the Frequency Domain

Spectrum of the sampled signal:

$$F_s(\omega_x, \omega_y) = \frac{1}{4\pi^2} F_I(\omega_x, \omega_y) * S(\omega_x, \omega_y)$$

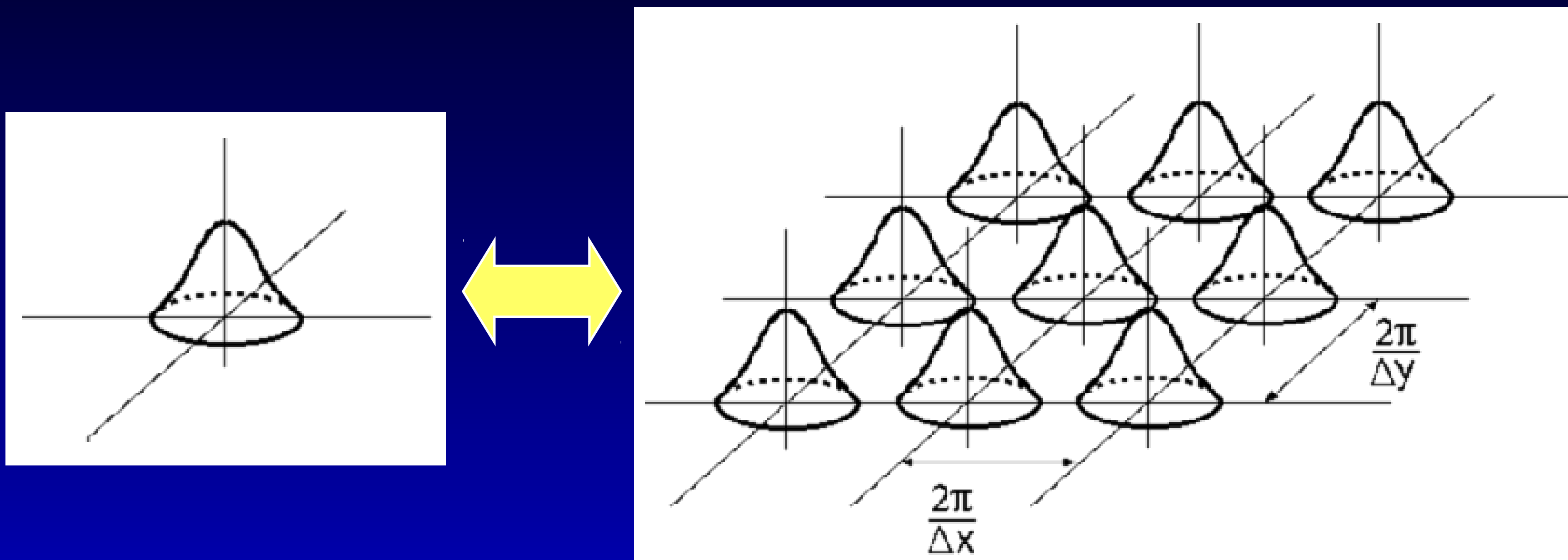
***S* (spectrum of the sampling function):**

$$S(\omega_x, \omega_y) = \frac{4\pi^2}{\Delta_x \Delta_y} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(\omega_x - i\omega_{xs}, \omega_y - i\omega_{ys})$$

and

$$\omega_{xs} = \frac{2\pi}{\Delta_x}, \omega_{ys} = \frac{2\pi}{\Delta_y}$$

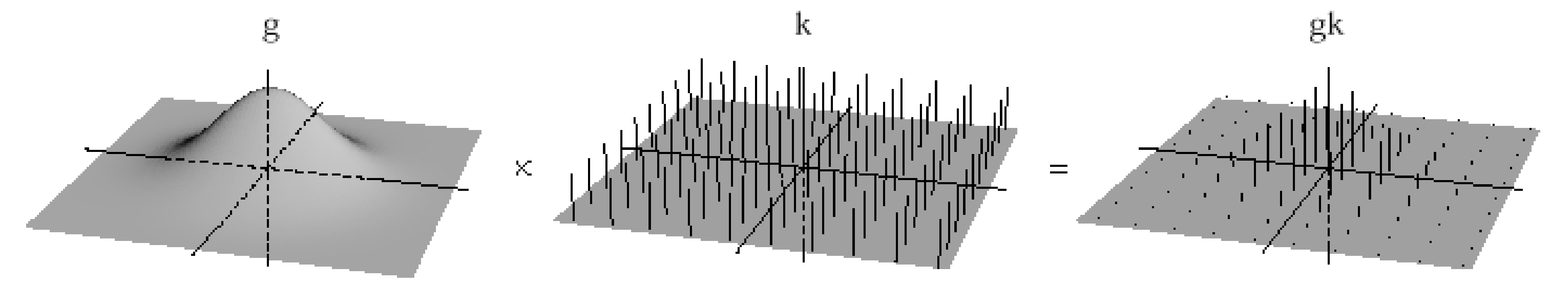
Sampling in the Frequency Domain



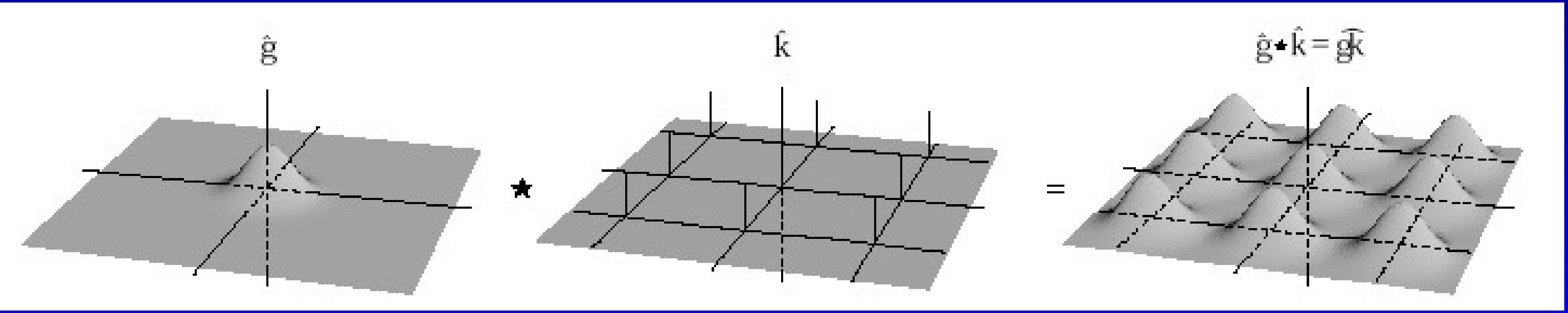
- Sampling a signal causes replication of its spectrum in the frequency domain
- Nyquist criterion:

$$\omega_s > 2\omega_m$$

Sampling of a Signal



In the spatial domain



In the frequency domain

Real-world Sampling Process

- The sampling function has a volume
- Can be approximated by a Gaussian:

$$s_r(x, y, z) = \exp\left(-\left(\frac{x^2}{2\sigma_x} + \frac{y^2}{2\sigma_y} + \frac{z^2}{2\sigma_z}\right)\right)$$

- A Gaussian is a low-pass filter, i.e. it causes blurred image, blunt edges

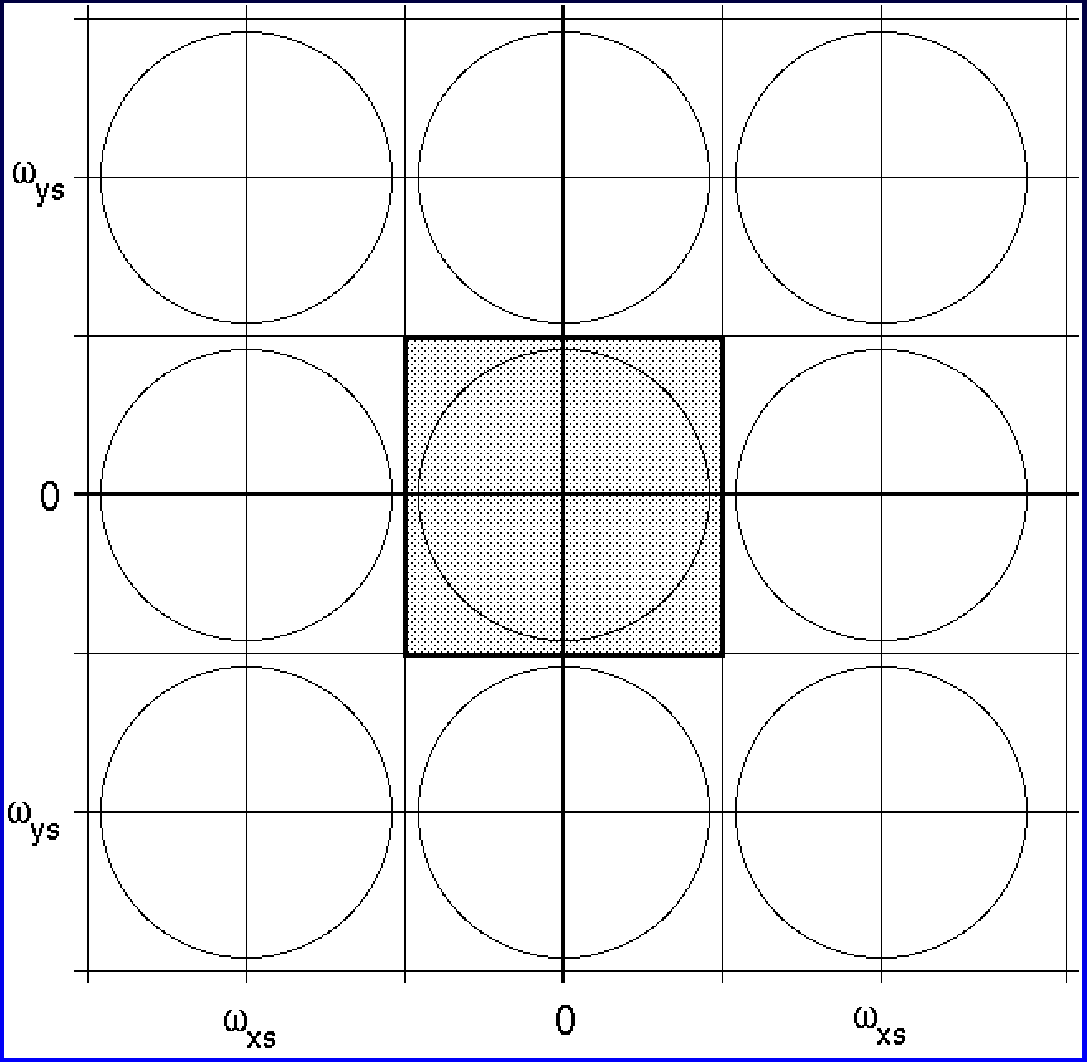
Reconstruction

Direct reconstruction of a continuous signal from discrete samples:

- The spectrum replicas have to be suppressed by a reconstruction filter
- The ideal reconstruction filter is the box function:

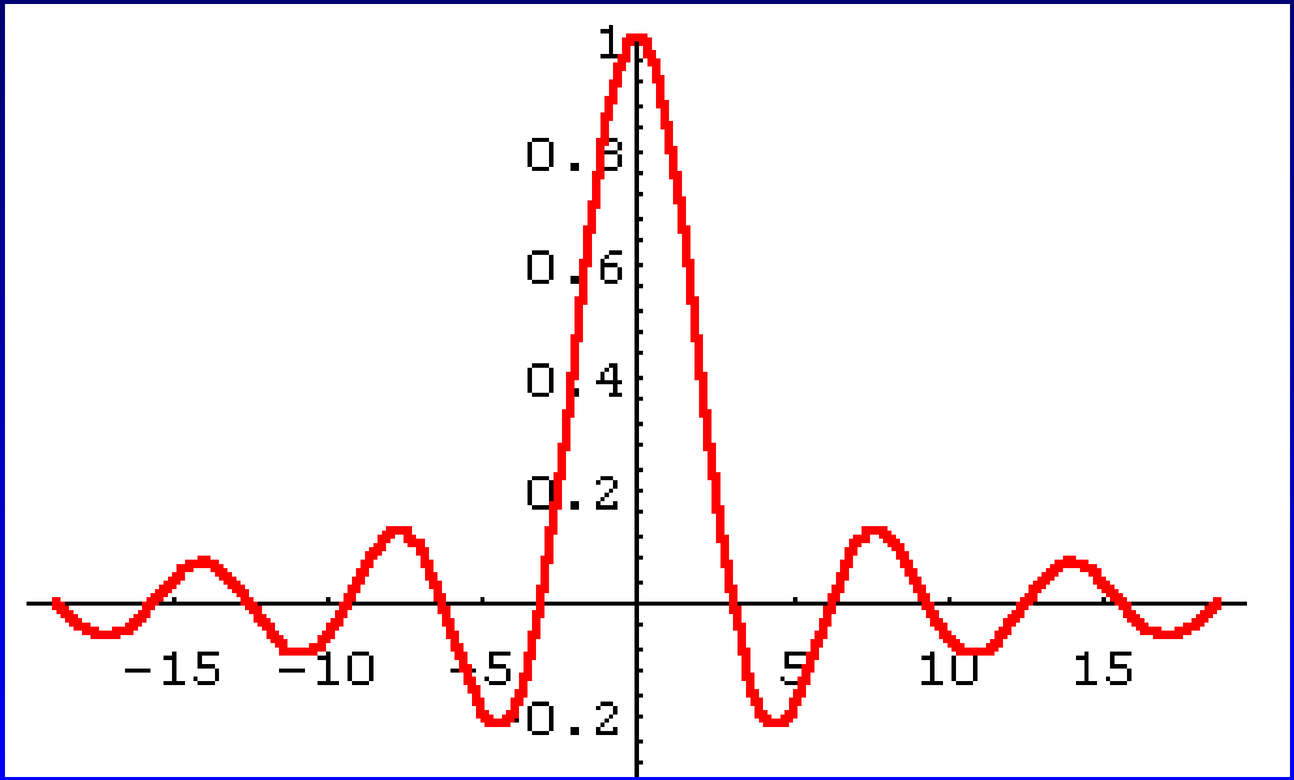
$$R(\omega_x, \omega_y) = \begin{cases} K \text{ if } |\omega_x| \leq \omega_{xc} \wedge |\omega_y| \leq \omega_{yc} \\ 0 \text{ else} \end{cases}$$

Reconstruction



Ideal Reconstruction Filter

$$\mathrm{box}(\omega) \leftrightarrow \mathrm{sinc}(s) = \frac{\sin(s)}{s}$$



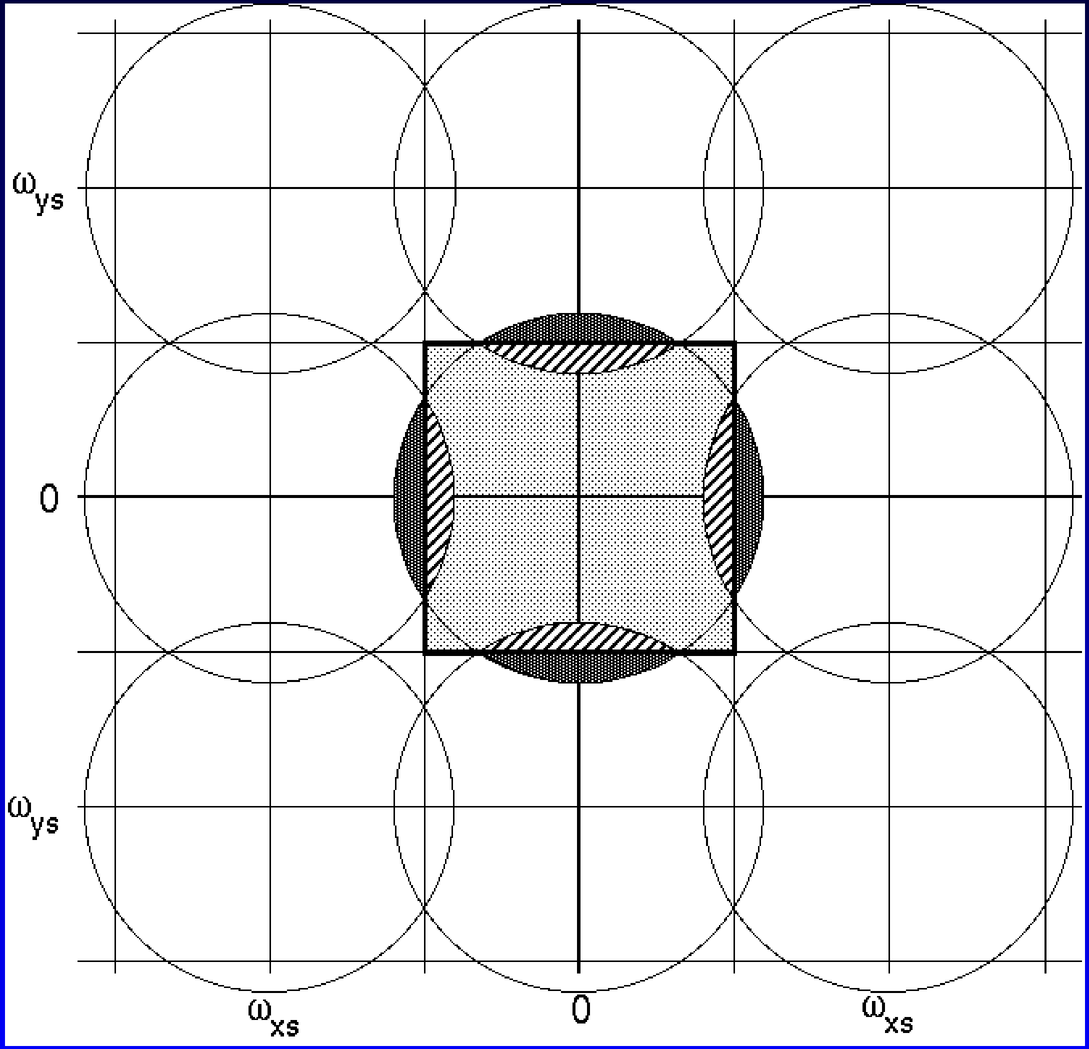
Ideal Reconstruction Filter

$$\mathrm{box}(\omega) \leftrightarrow \mathrm{sinc}(s) = \frac{\sin(s)}{s}$$

- Bounded support in the frequency domain, but unbounded in the spatial domain
- Can’t be realized practically
- It has to be approximated (bounded):
  - The approximations have unbounded support in the frequency domain

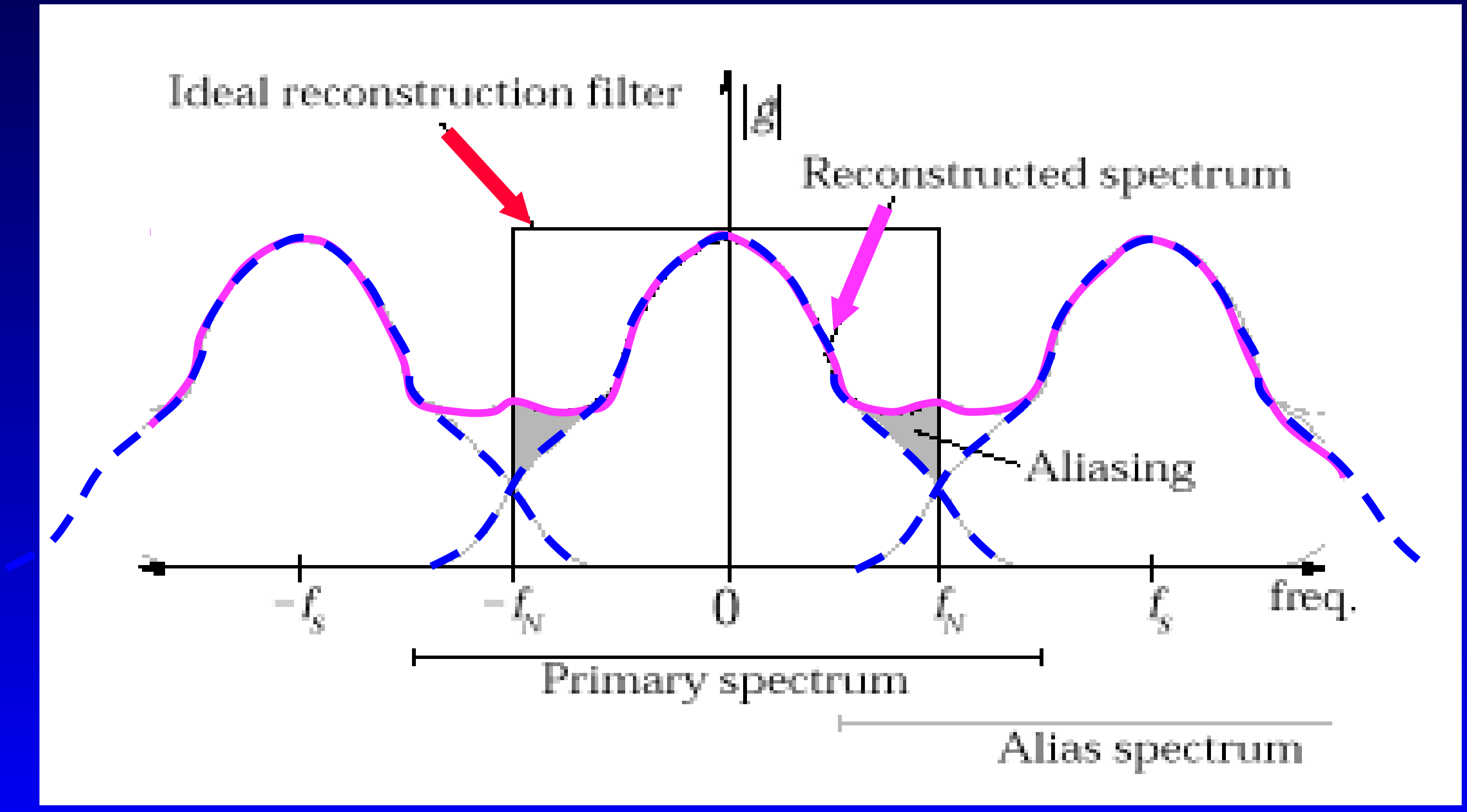
Problems with the Reconstruction (aliasing)

- If the Nyquist criterion is not fulfilled
- If the reconstruction filter is too big or too small



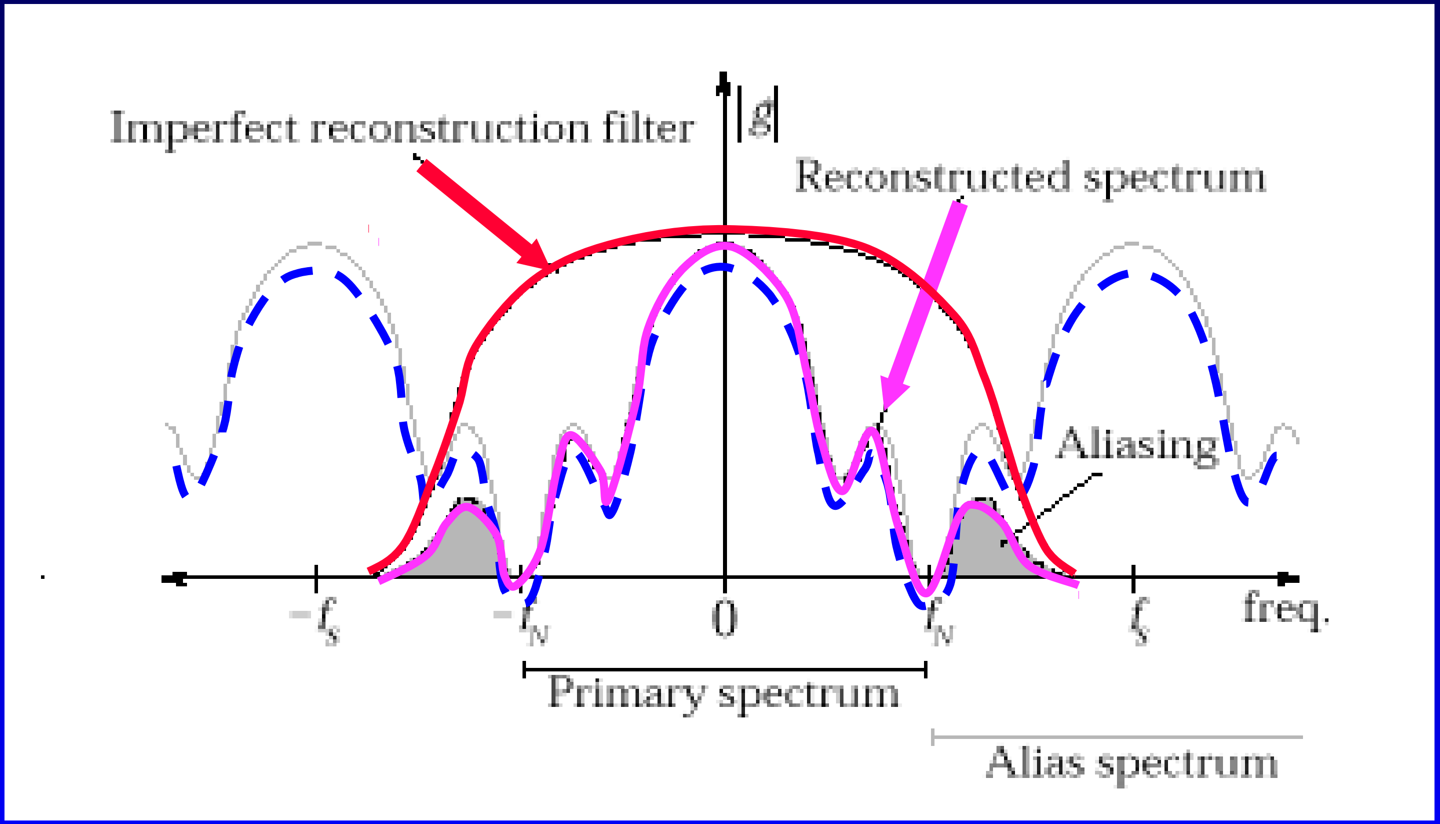
Prealiasing

- Wrong frequencies appear if the Nyquist criterion is not fulfilled:



Postaliasing

- Wrong frequencies appear also if the reconstruction filter support is too broad



Classification of 3D reconstruction filters

- Separable:

$$h(x, y, z) = h_s(x)\ h_s(y)\ h_s(z)$$

  - Sequential application along the axes
  - Computational complexity: *3n*
- Spherically symmetrical:
  - Computational complexity: n^3

Separable Filters

- Order 0: nearest neighbor

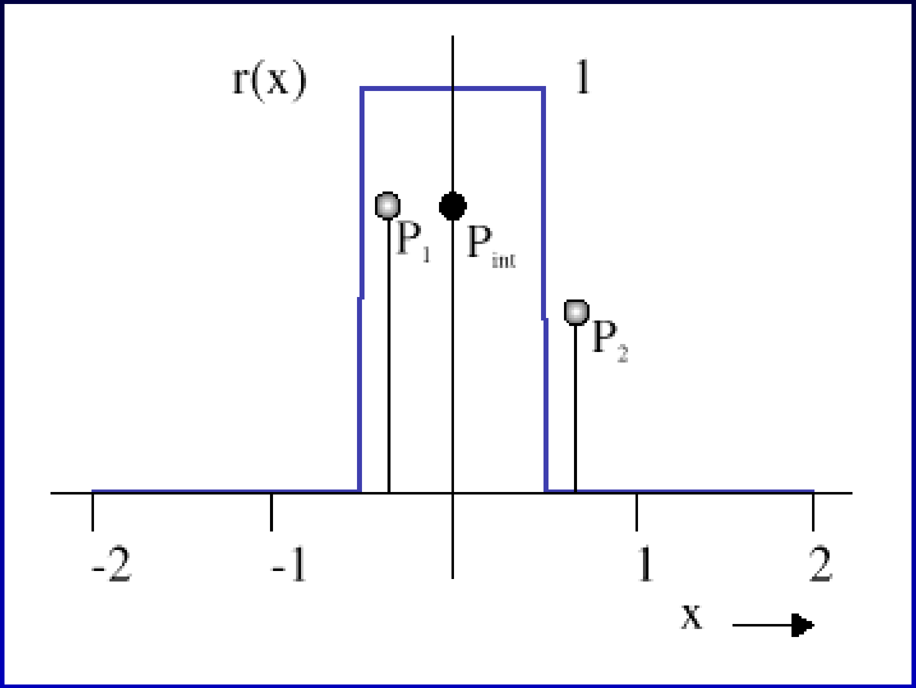
$$h_s = 1 \text{ if } |x| < 0.5$$

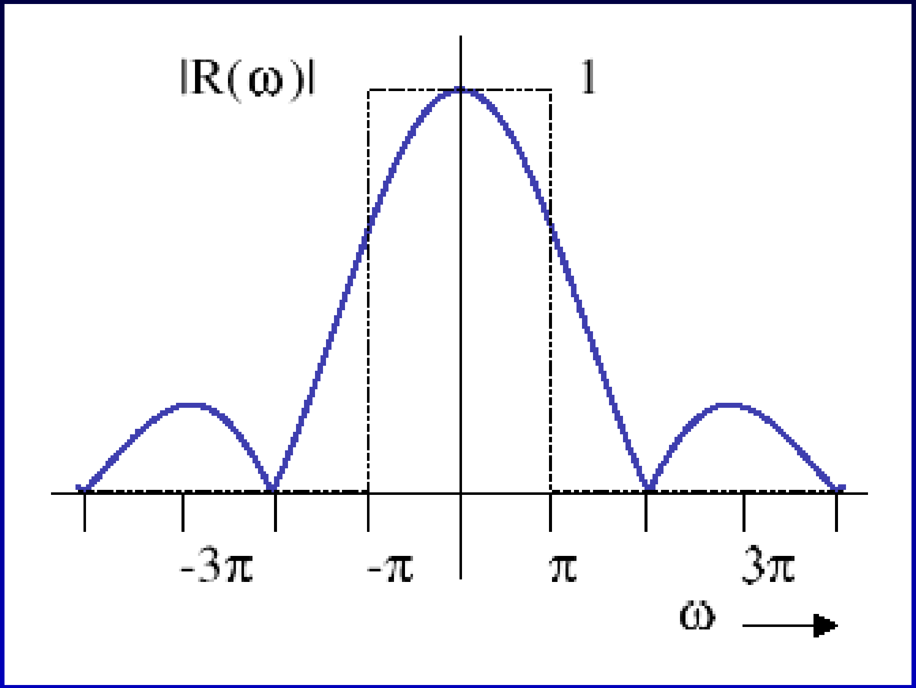
- Order 1: linear interpolation

$$h_s = 1 - |x| \text{ if } |x| < 1$$

- Order 3: cubic filters:
  - Cubic B-spline
  - Catmull-Rom spline

Nearest Neighbor

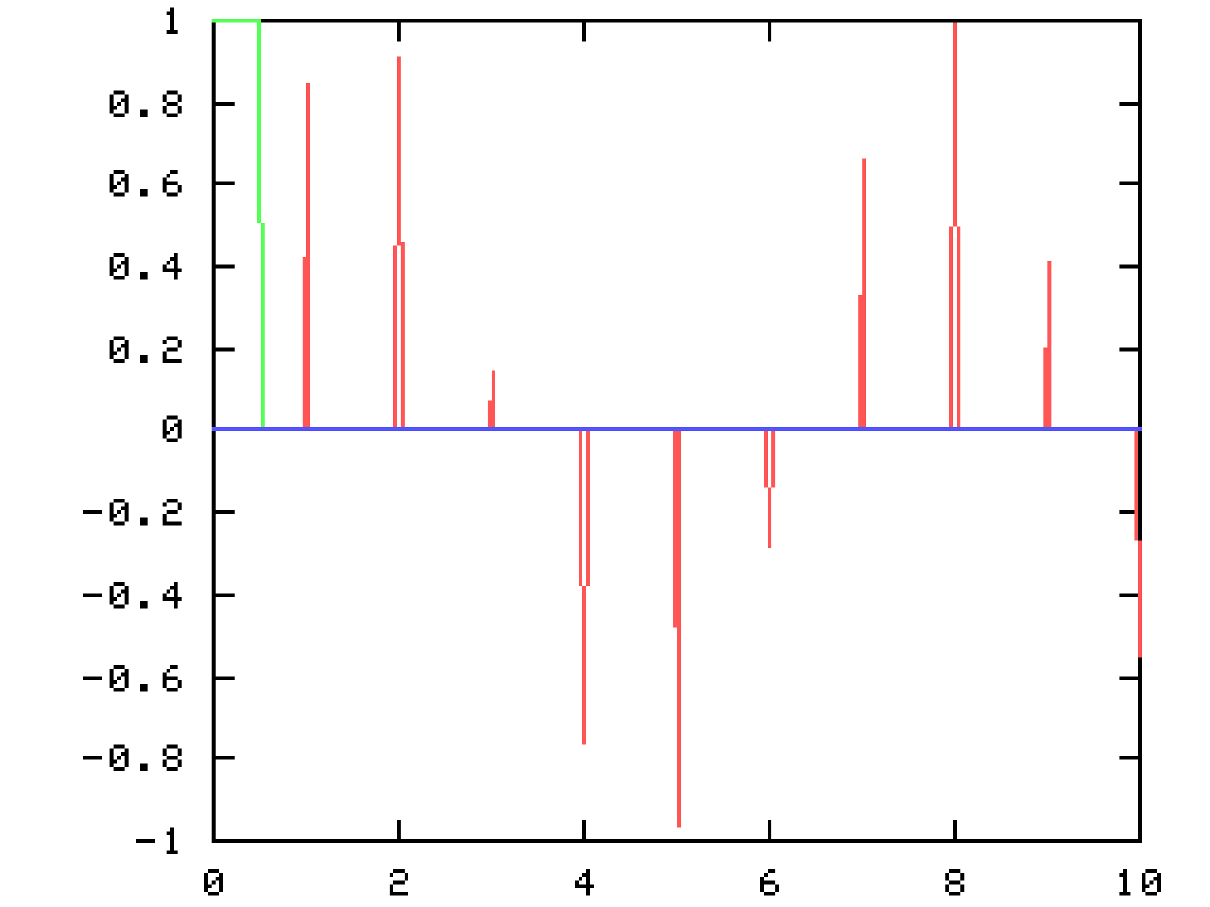




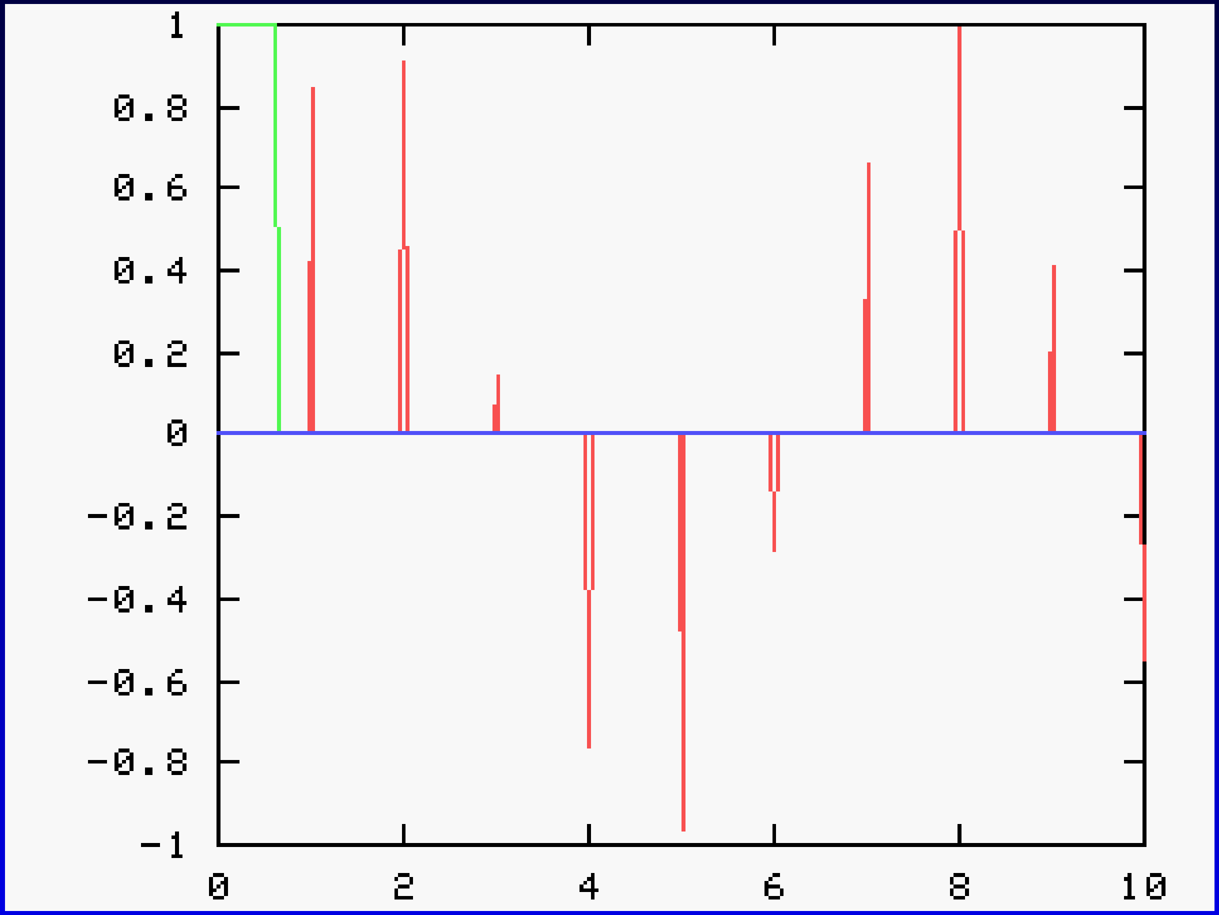
Filter

Spectrum

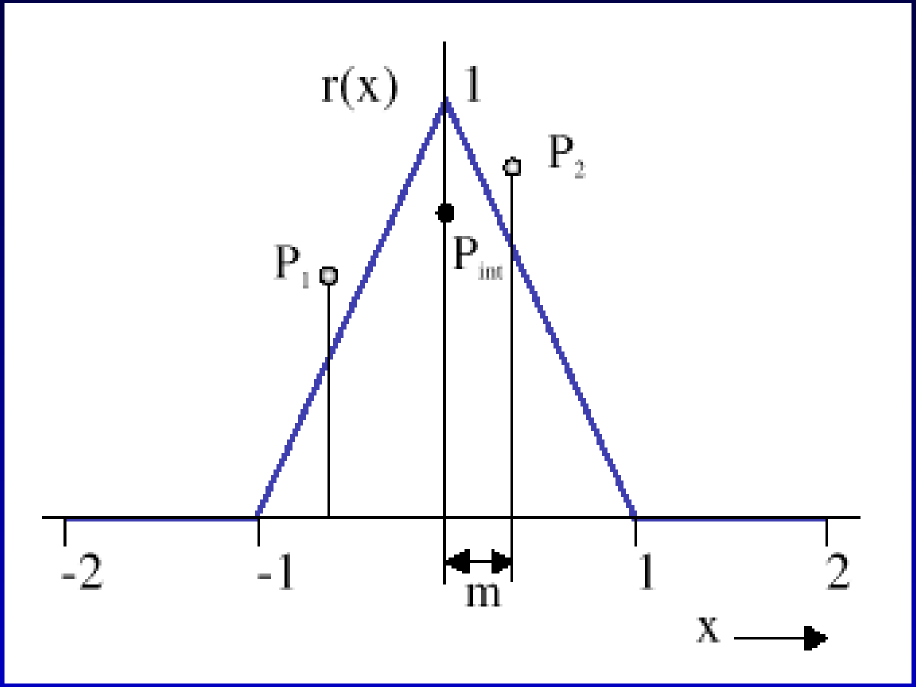
Nearest Neighbor Filtering

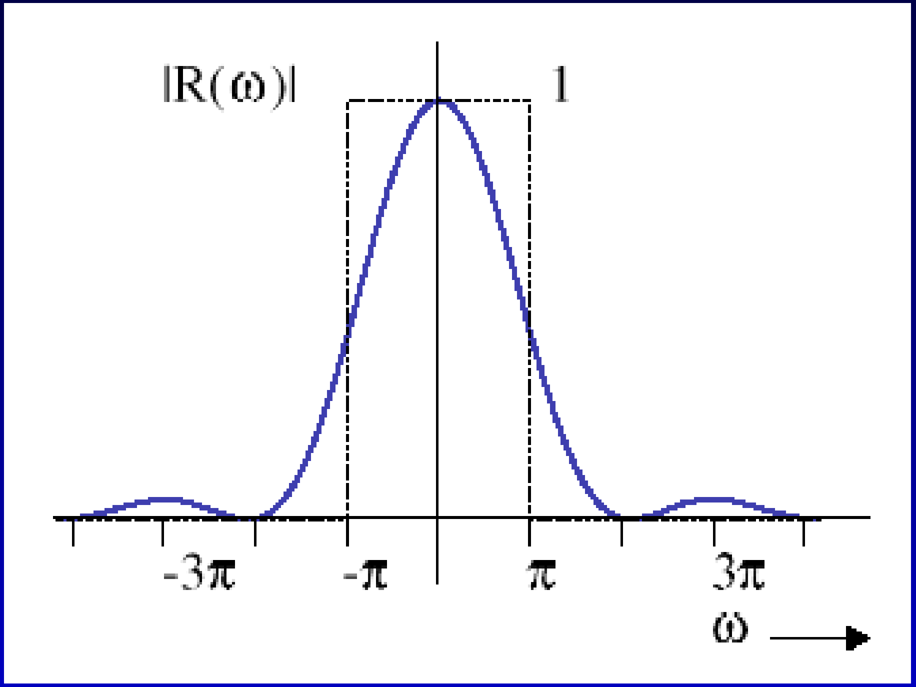


Nearest Neighbor Filtering



Linear Interpolation

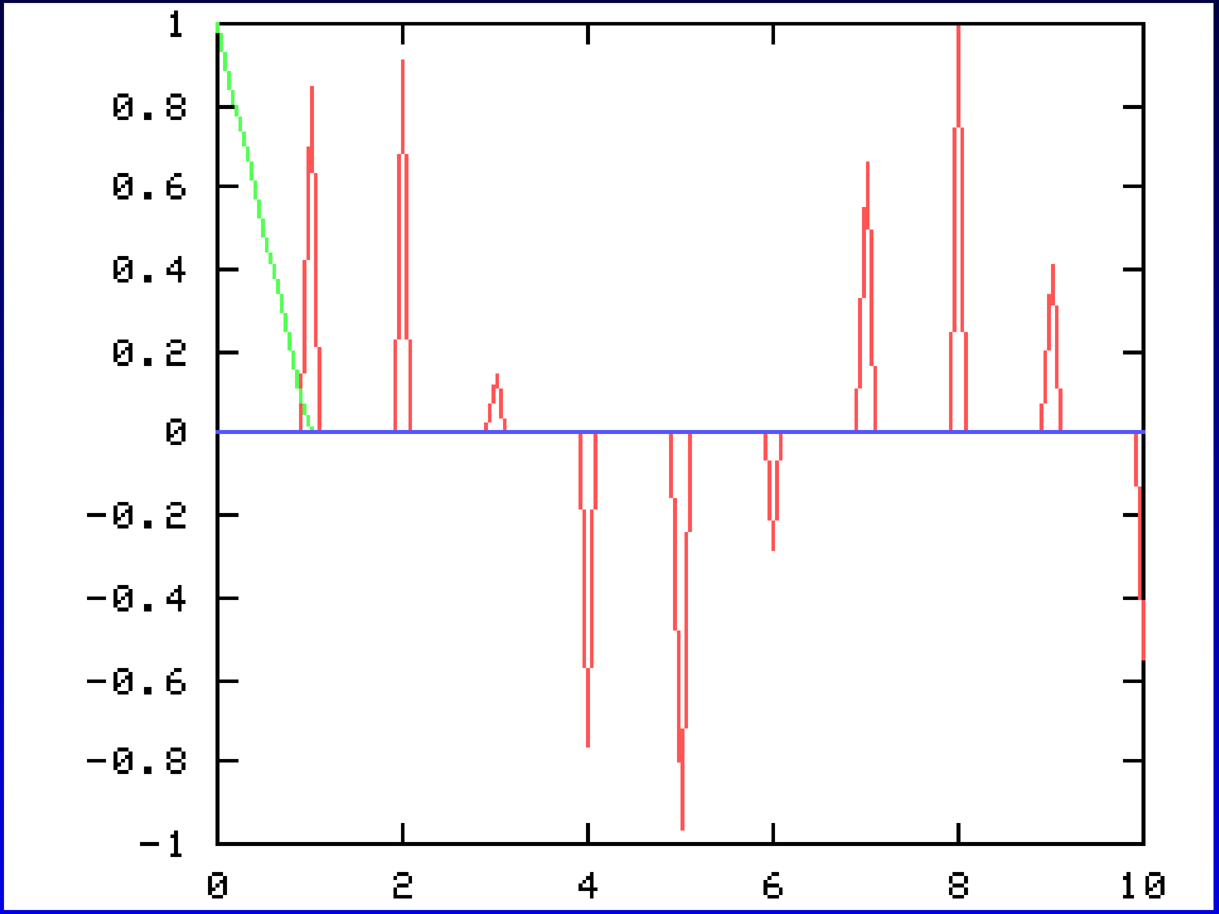




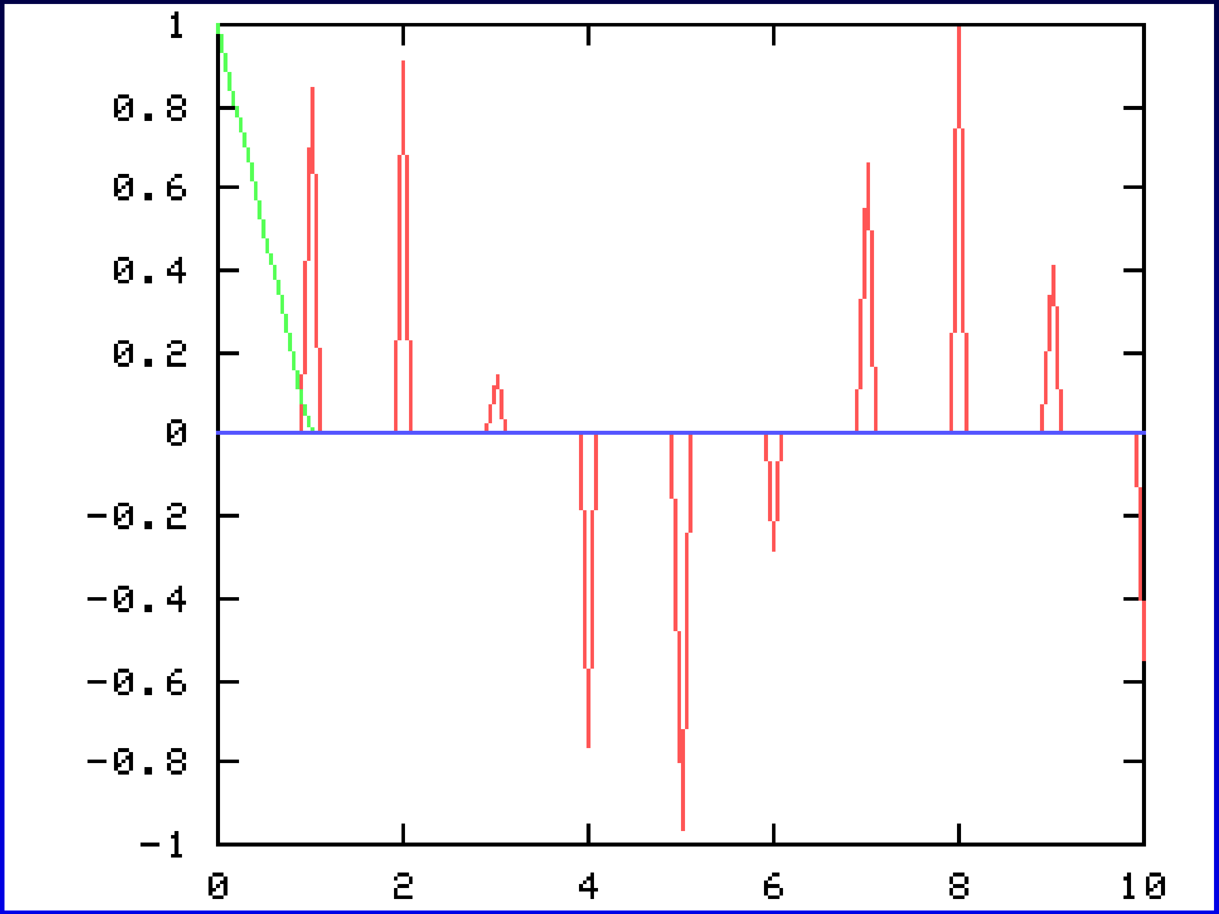
Filter

Spectrum

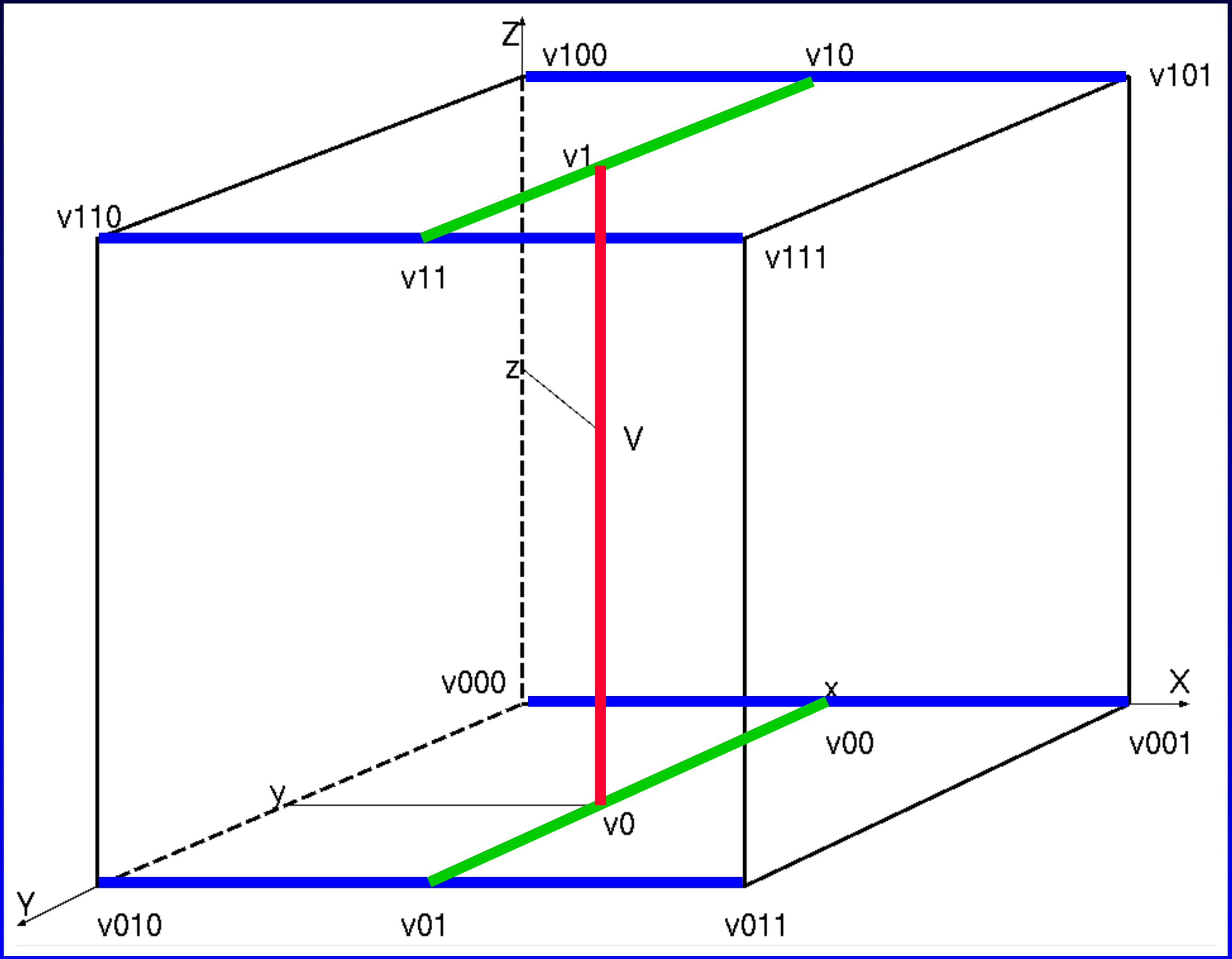
Linear Filtering



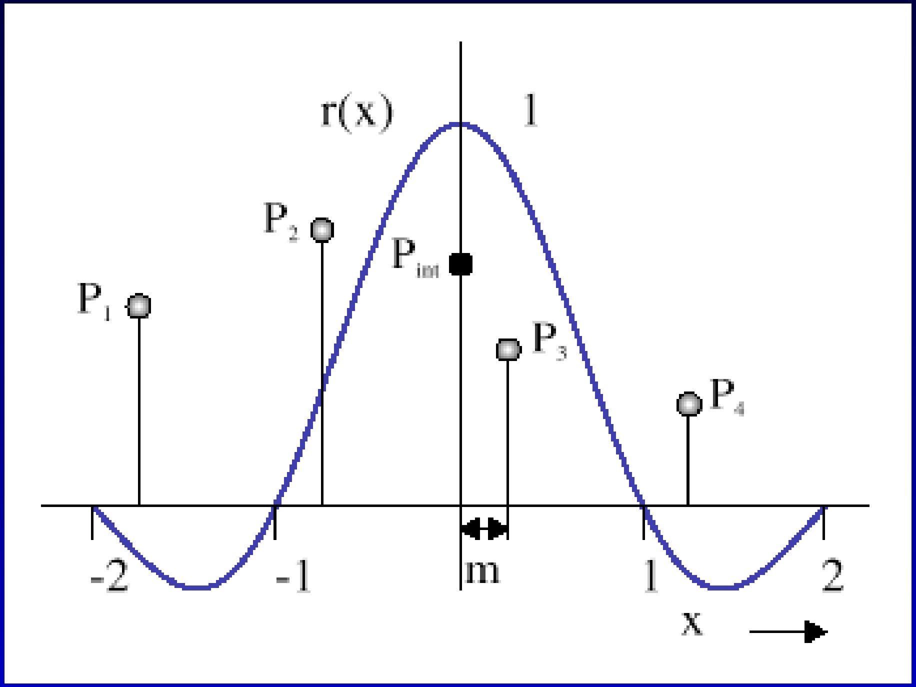
Linear Filtering



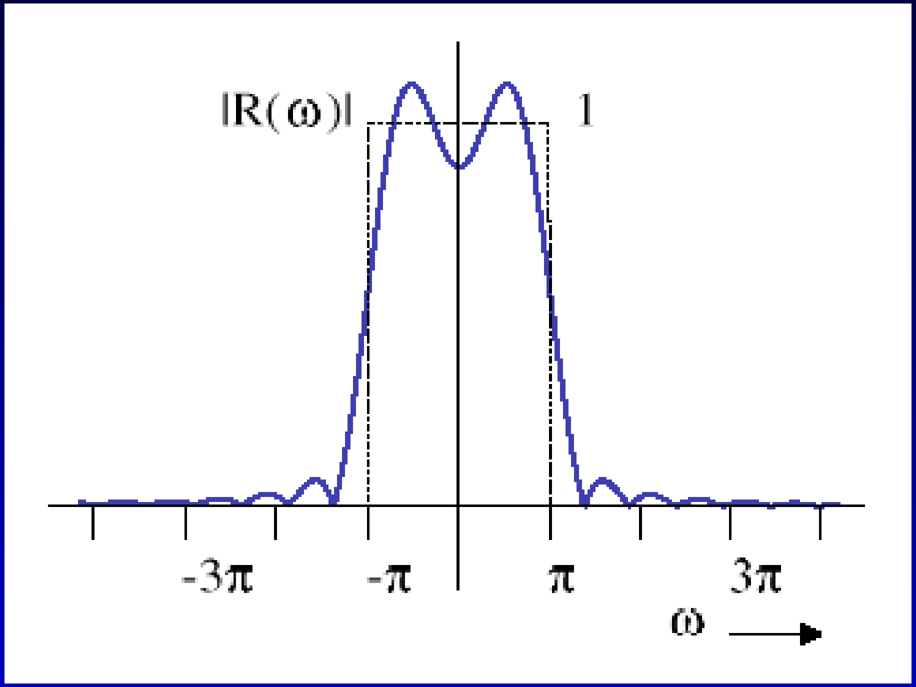
Trilinear Reconstruction Filter



Truncated *sinc* Filter

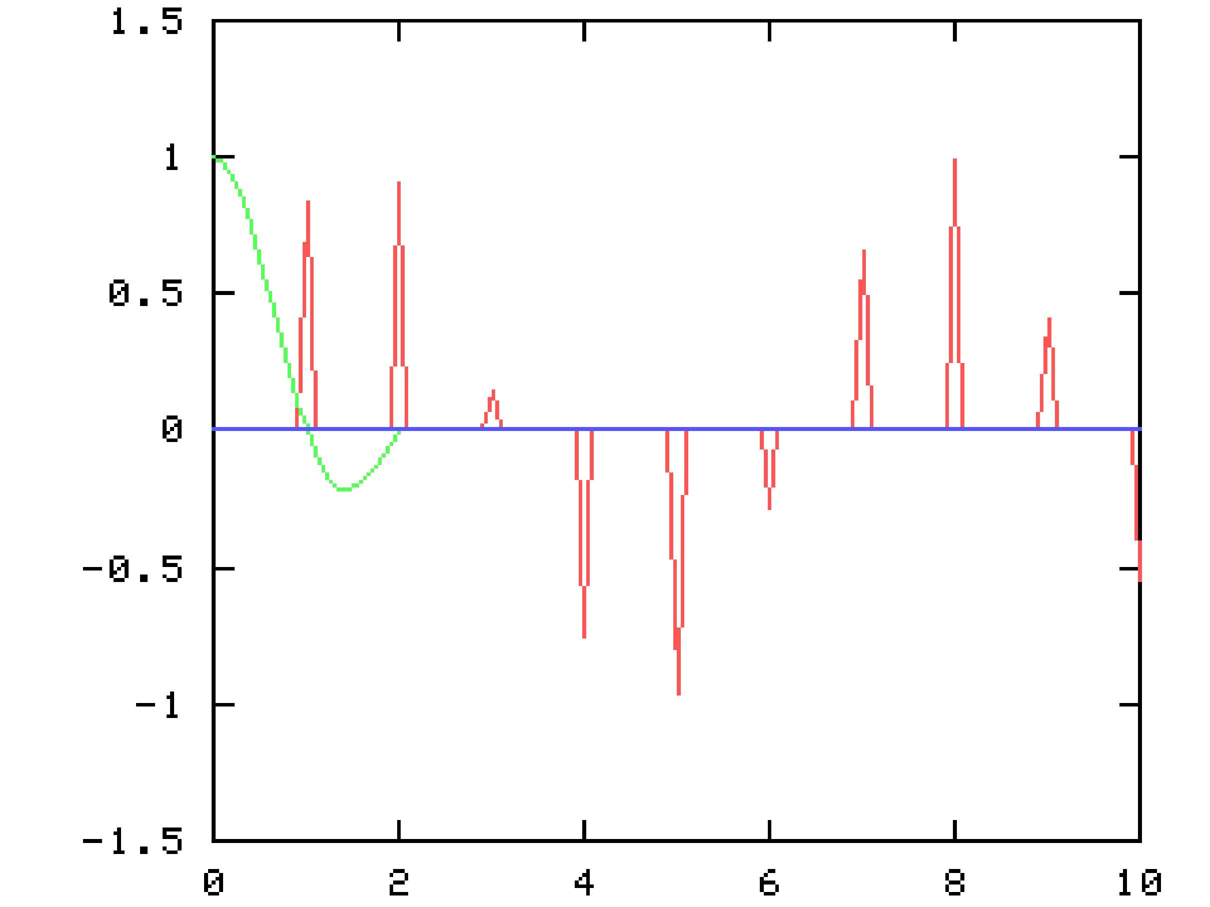


Filter

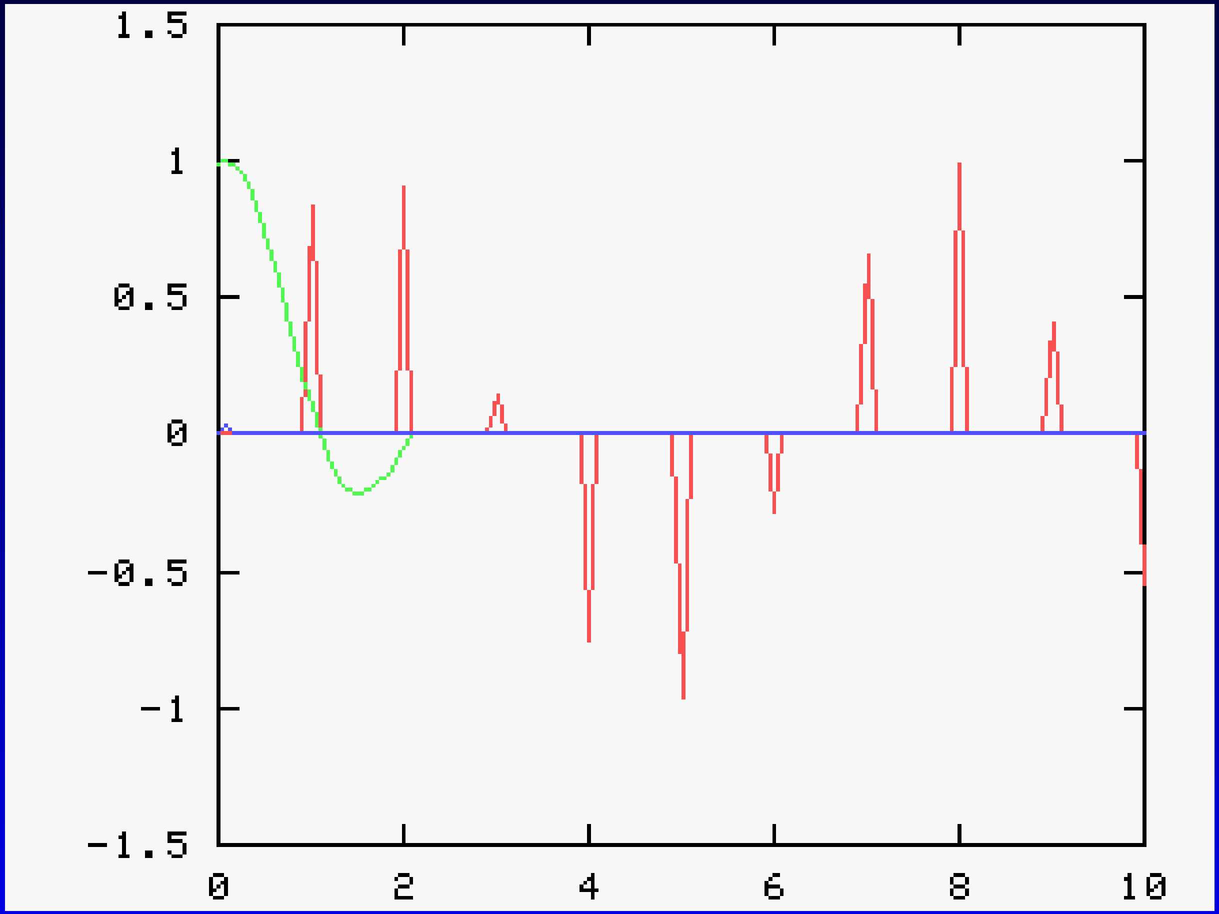


Spektrum

Truncated *sinc* Filtering

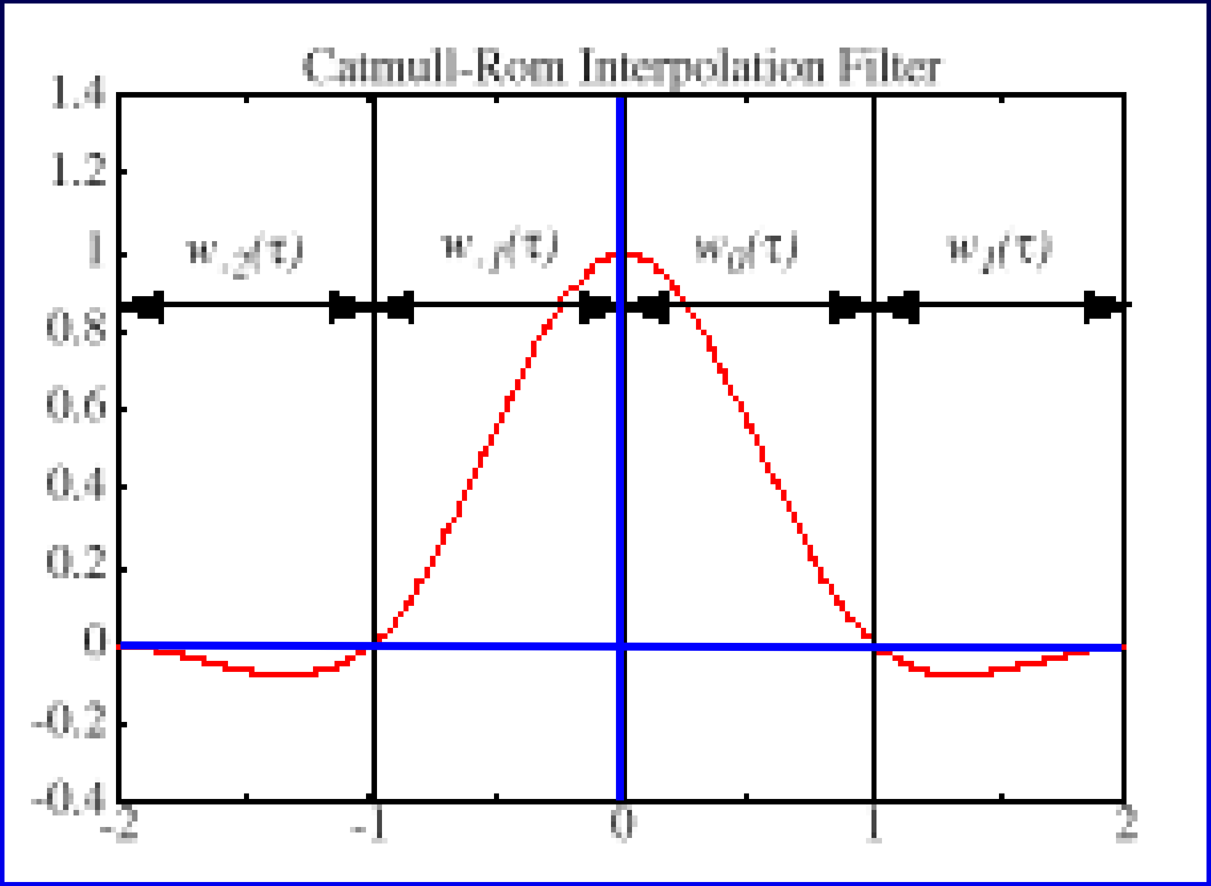


Truncated *sinc* Filtering

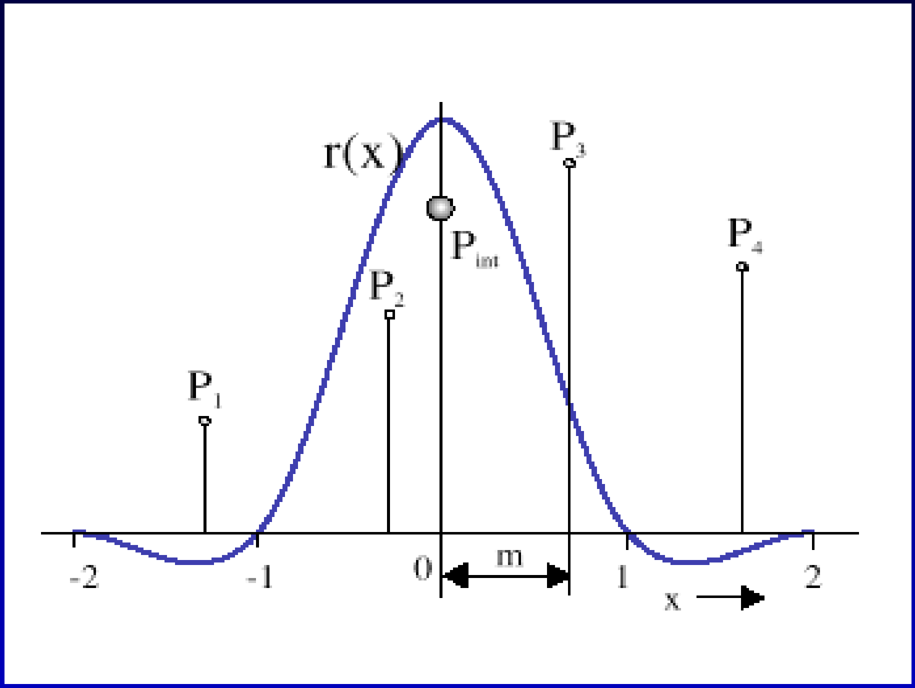


Catmull-Rom Spline

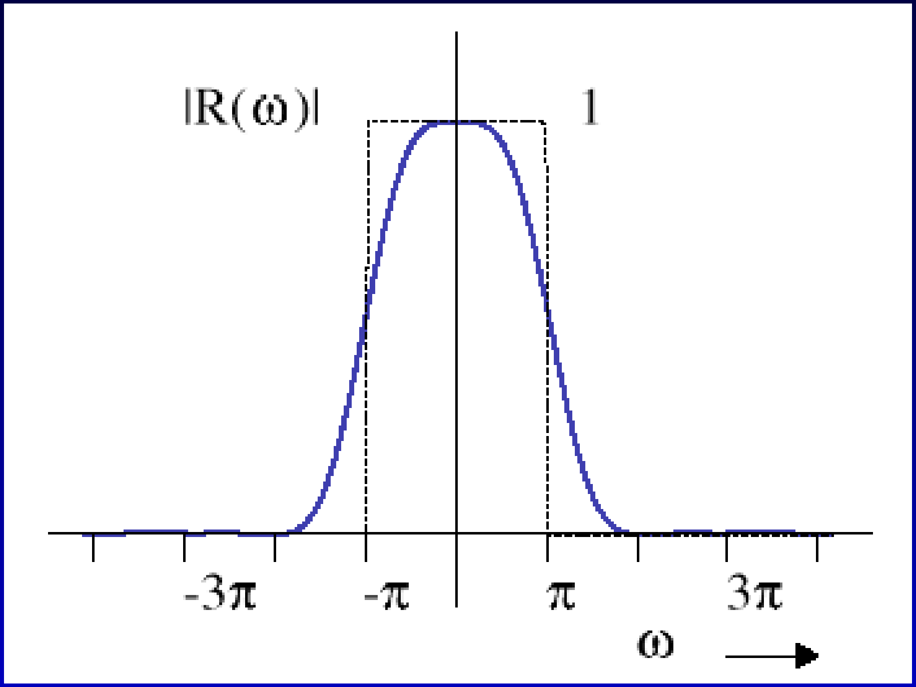
- Piecewise cubic, C1-smooth



Catmul-Rom Spline

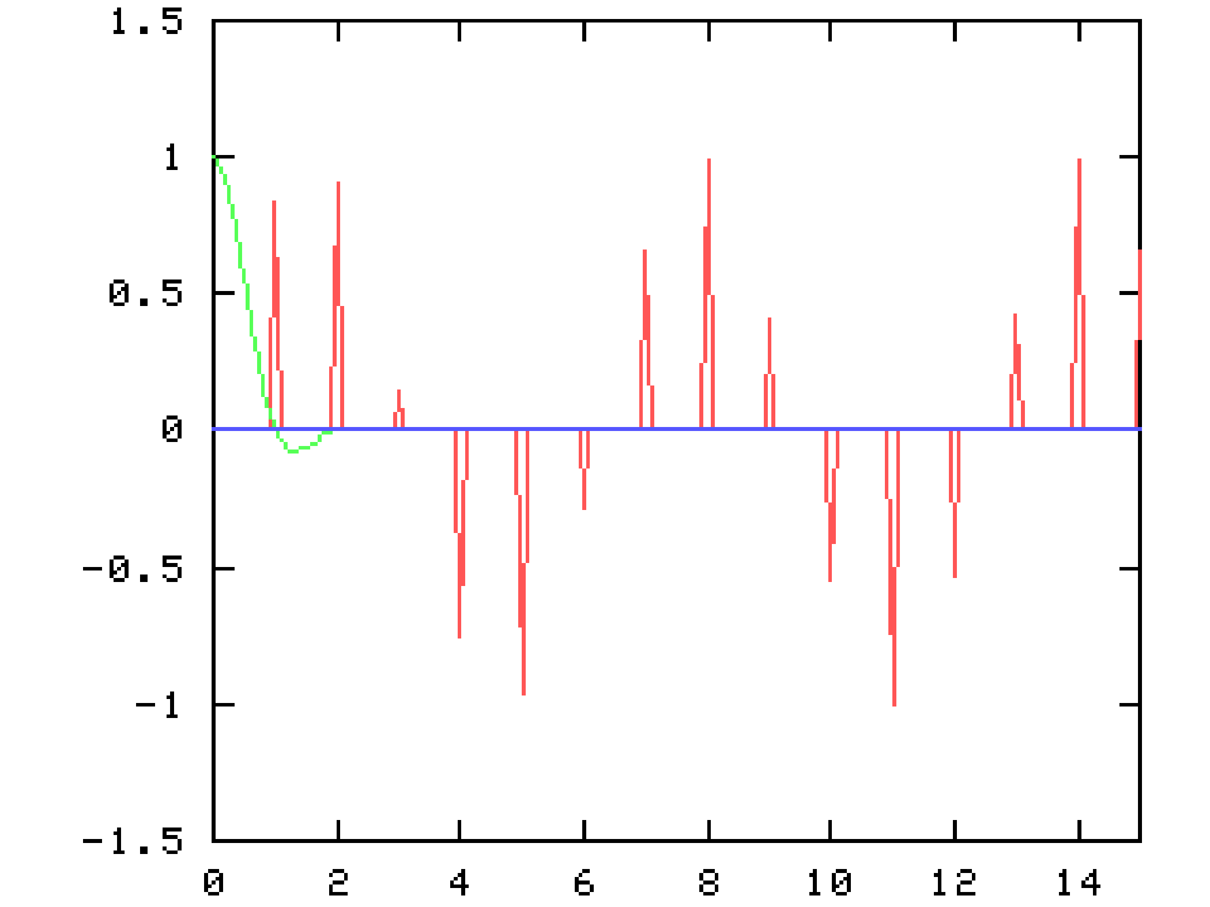


Filter

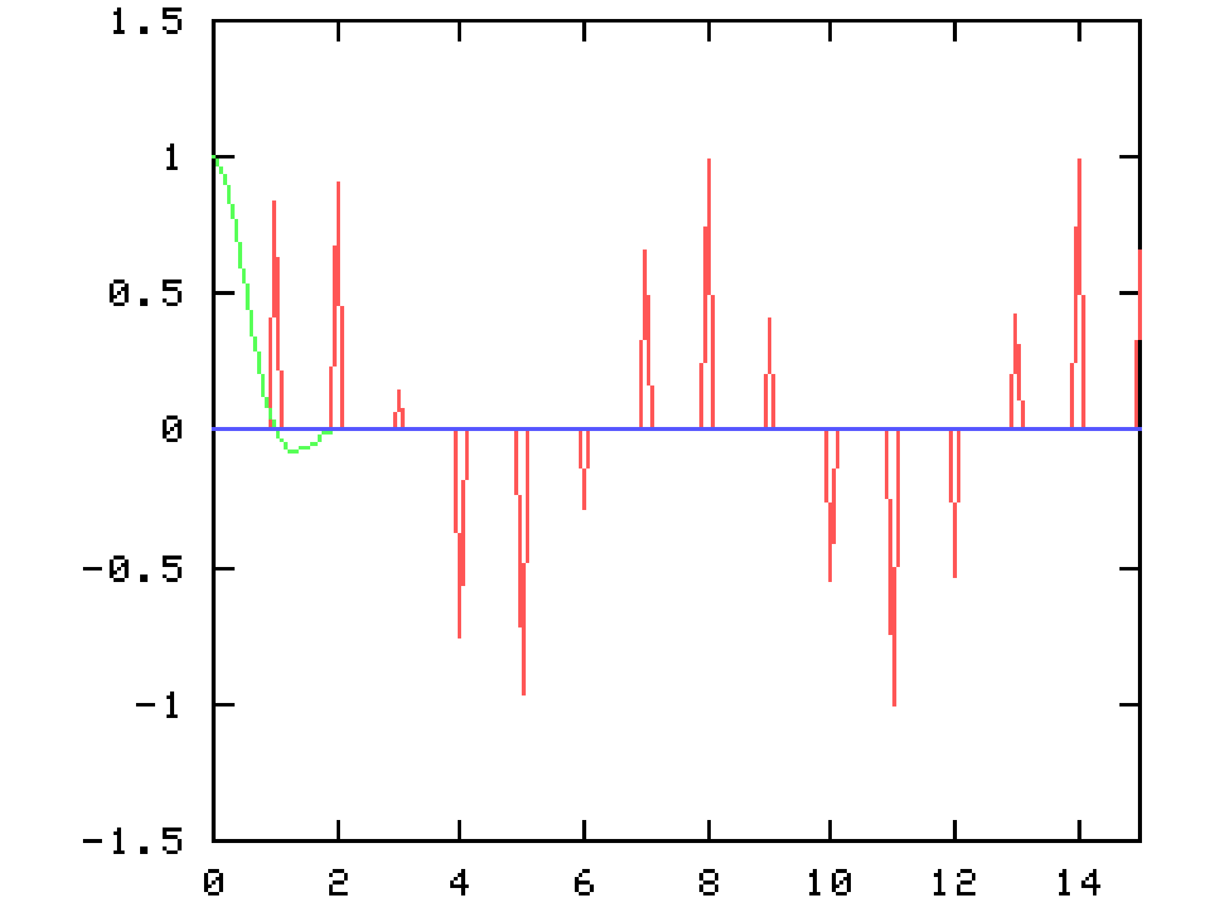


Spectrum

Catmull-Rom Spline Filtering



Catmull-Rom Spline Filtering



Other Separable Filters

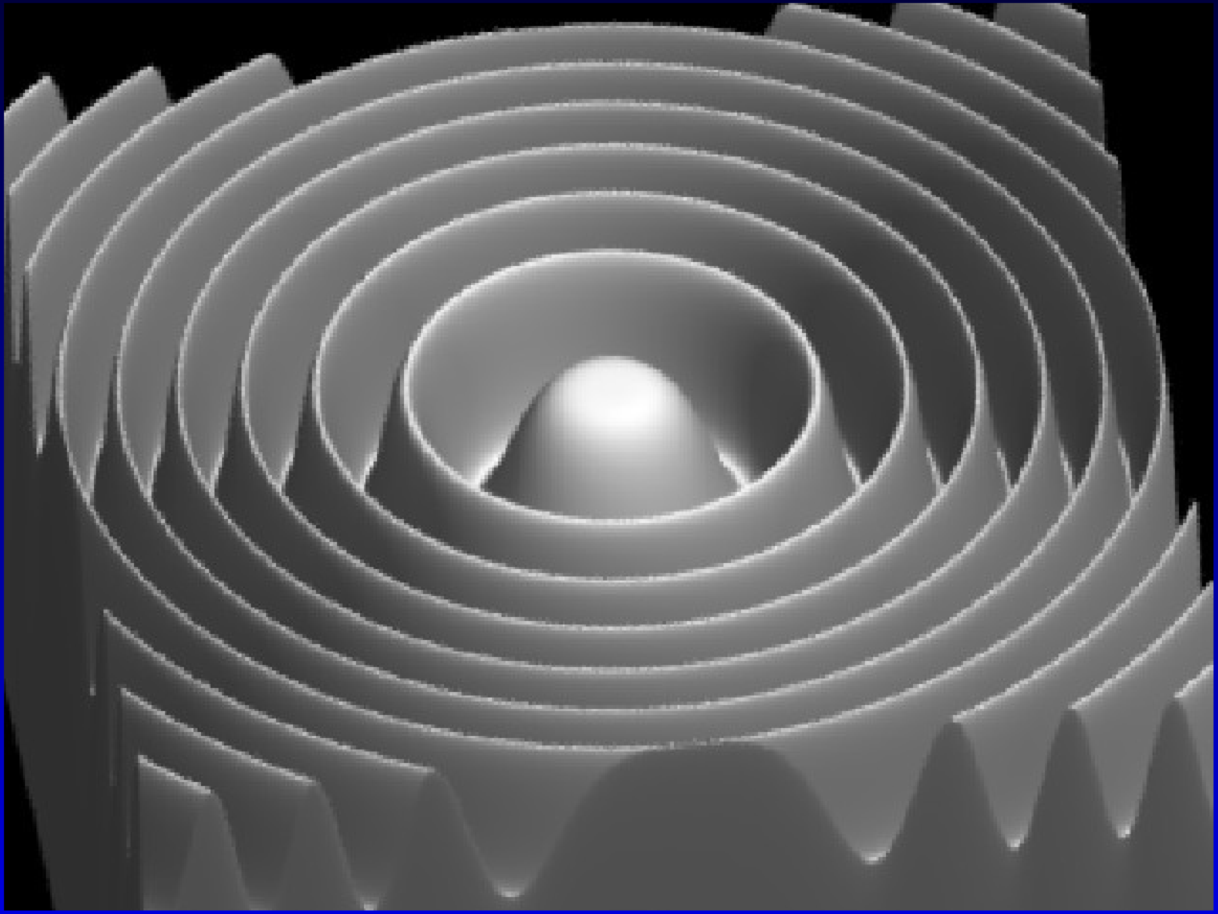
- Gaussian filter

$$h_s(x) = \exp(-\frac{x^2}{2\sigma^2}), \quad |x| < x_m$$

- Windowed *sinc* filter

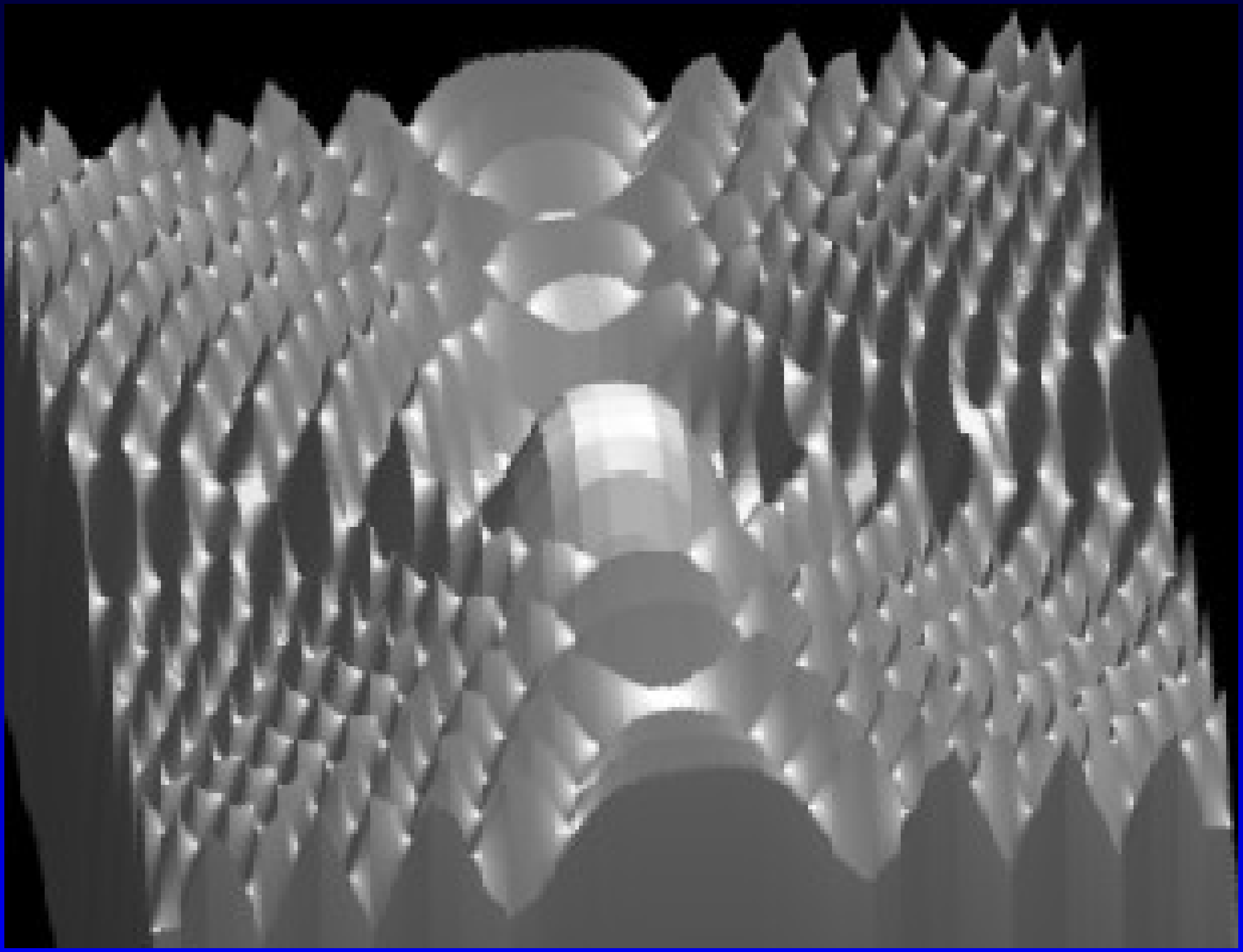
$$h_s(x) = \left(1 + \cos(\frac{\pi x}{x_m})\right) \mathrm{sinc}(\frac{4x}{x_m}), \quad |x| < x_m$$

An Example

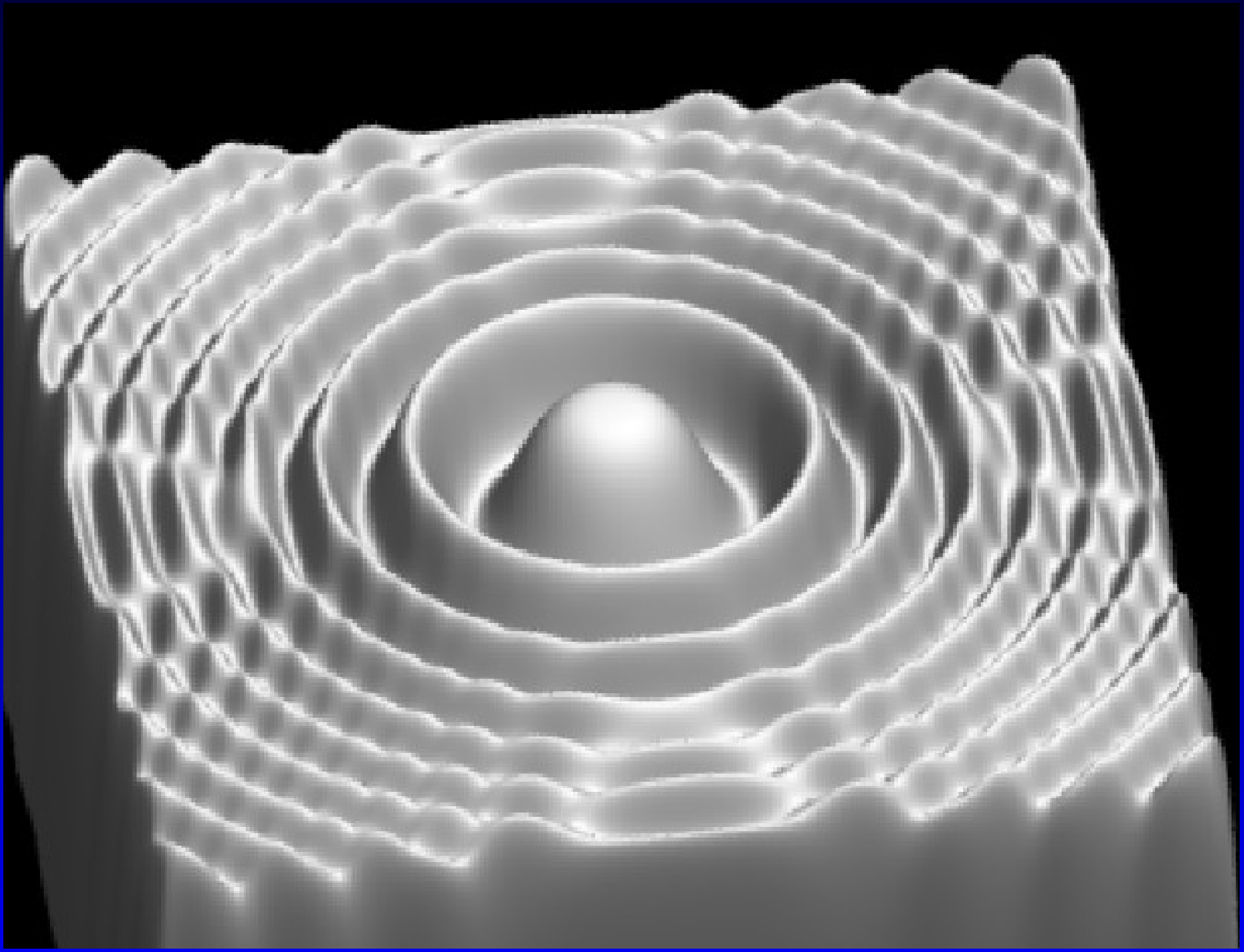


$F_N = 10$, sampling 40 x 40 x 40

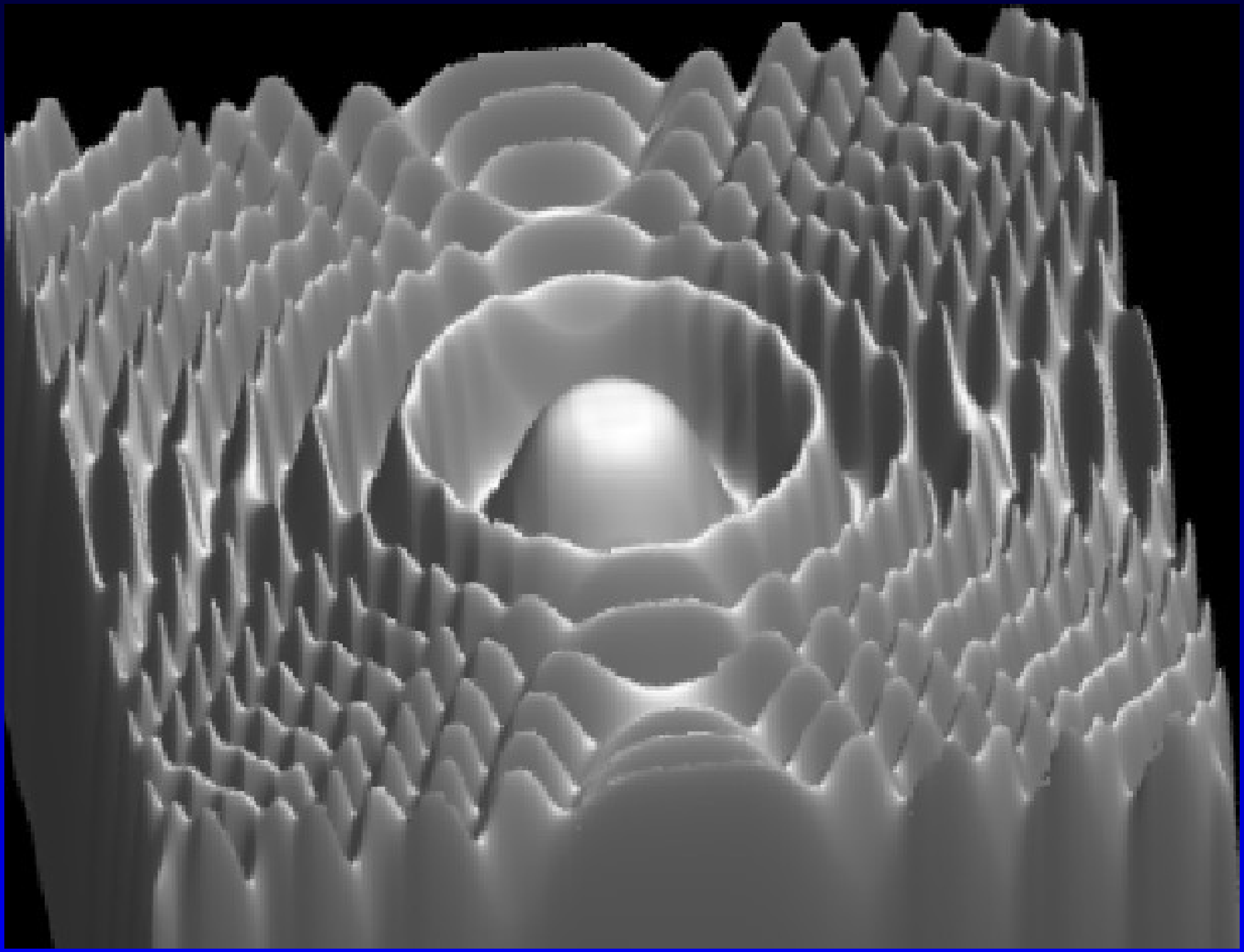
Reconstruction by the Trilinear Filter



Reconstruction by the Cubic B-spline Filter



Reconstruction by the Catmull-Rom Spline Filter



Reconstruction by the Windowed *sinc* Filter