7. Stochastic Fractals

- Simulation of Brownian motion
- Modelling of natural phenomena, like terrains, clouds, waves, ...
- Modelling of microstructures, like erodism, fur, bark, ...
- Usefull for solid textures, like marble, ...

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Diffusion Limited Aggregation (DLA)

- Electrodeposition of zinc sulfate particles

![Diagram showing electrodeposition process](image1)

- Carbon cathode
- Zinc anode
- n-butyl-acetate
- Zinc sulfate solution

Diffusion Limited Aggregation (DLA)

![Fractal pattern](image2)
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Diffusion Limited Aggregation (DLA)

- Particles are suspended in a liquid and walk randomly towards the cathode
- Such an erratic movement of particles in a liquid is described by Brownian motion
- The effect is caused by very light collisions with surrounding molecules
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**Diffusion Limited Aggregation (DLA)**

- Complex field lines between crystal branches also influence the dynamic

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**Lichens – Aggregation of Microorganisms**

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7.4
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Simulation of DLA

- Using a pixel raster, one pixel is the cathode
- Define a region of interest around the cathode pixel, e.g. a circle
- Inject a particle at the boundary of the region and let it move randomly from one pixel to another
- If it walks out of the region, forget it and start again with a new particle

If it gets close to the cathode it attaches and becomes an additional cathode pixel

The procedure is repeated until a dendrite evolves
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Simulation of DLA
- Aggregation time determines color

3D-DLA
- Website:
  - markjstock.org/dla3d/

7.6
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Brownian Motion

- Discovered 1827 by the botanist R. Brown
- Describes the movement of small particles of solid matter in liquid
- Mathematical examination by Einstein and Wiener
- Used by Mandelbrot for modelling of natural phenomena

Statistical Basics

- Continuous random variable $X$
  - Density function $f(x) \geq 0$: $\frac{d}{dx} F(x)$
  - Distribution function $F(x) = P(x \leq X): \int_{-\infty}^{x} f(t) dt$
  - Expectation $E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$
  - Varianz $\text{Var}[X] = E[(X-E[X])^2]$
  - Standard deviation: $S(n) = \sqrt{\text{Var}[X]}$
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Statistical Basics

- Correlation: Measurement for the mutual dependency of two random variables
- Some examples:
  - Gaussian distribution $N(\mu, \sigma)$
  - Normal distribution $N(0, 1)$
  - Exponential distribution
  - Bernoulli distribution
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Statistical Basics

Density function
Gaussian distribution

Brownian Motion

- The position $X(t)$ of a particle at time $t$ is the result of a stochastic (random) process:
  - $X(t+\Delta t) = X(t) + v \cdot \Delta t^{0.5} \cdot N(0,1)$
  - $v$ ... average speed of a particle
  - $N(0,1)$ ... normal distributed random variable
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Brownian Motion

- Properties:
  - The increment $X(t_2)-X(t_1)$ has Gaussian distribution $\Rightarrow E[X(t_2)-X(t_1)] = 0$
  - $\text{Var}[X(t_2)-X(t_1)] \propto |t_2-t_1|$
  - Continuous but not differentiable
  - The increments $X(t_0+t)-X(t_0)$ and $1/r^{0.5} (X(t_0+rt)-X(t_0))$ are statistically self similar

- If $t_0 = 0$, $X(t_0) = 0 \Rightarrow X(t)$ and $X(rt)/r^{0.5}$ are statistically equivalent

$r=8$
$r=4$
$r=2$
$r=1$
$r=0.5$
$r=0.25$
$r=0.125$
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White Noise

- White noise, also called $1/f^0$-noise is completely uncorrelated from point to point.
- Its spectral density is a flat line $\Rightarrow$ equal amounts at all frequencies, like white light.
- Simulated by pseudo random generator.

\[
\begin{align*}
\log S(f) & \quad 1/f^0 \\
\log f & \quad \log f
\end{align*}
\]

1/f-Noise

- Its physical origin is still a mystery.
- It is correlated from point to point.
- Most common type of noise found in nature (ocean flows, nerve membranes,...).
- No simple model to produce it.

\[
\begin{align*}
\log S(f) & \quad 1/f \\
\log f & \quad \log f
\end{align*}
\]
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### Brownian Motion
- It is the integral of white noise, also called $1/f^2$-noise
- Is more correlated than $1/f^0$- and $1/f$-noise
- It consists of many more low frequencies than high frequencies

![Brownian Motion Graph](image)

### The Hurst Exponent
- The Hurst Exponent $H$ was developed for hydrology by Harold Edwin Hurst
  - Study of time series of river Nile floodings
  - Allows better planning of dam sizes
- Assess variability of time series depending on observation period(s)
- Measure for the autocorrelation (or long term memory) of time series
  - $\Rightarrow$ measure for stochastic self-similarity
- Directly related to the fractal dimension
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The Hurst Exponent

Definition:

\[ E \left[ \frac{R(n)}{S(n)} \right] = Cn^H, n \to \infty \]

Estimation (similar to box counting):

1. Divide time series \( X \) of length \( n \) into subsets with length \( n_1=n/2, n_2=n/4, \ldots \)
2. Calculate the averaged rescaled range \( R/s \) for all divisions
3. Plot log/log diagram of \( E[R/s] \) vs. \( n_i \)
4. Slope of fitting line is \( H \)
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The Hurst Exponent

Calculating the rescaled range:
1. Create mean adjusted time series:
   \[ Y_t = X_t - E[X], \ t=1,2,...,n \]
2. Calculate cumulative deviate series:
   \[ Z_t = \sum_{i=1}^{t} Y_i, t=1,2,...,n \]
3. The range \( R(n) \) is given by:
   \[ R(n) = \max(Z_1, Z_2, ..., Z_n) - \min(Z_1, Z_2, ..., Z_n) \]
4. Rescaled range = \( R(n) / S(n) \), whereby \( S(n) \) is the standard deviation of the range

Fractal Brownian Motion (FBm)

Extension of Brownian motion using the Hurst Exponent \( H \) as parameter

Properties:
- \( \text{Var}[X(t_2)-X(t_1)] \propto |t_2-t_1|^{2H} \)
- \( X(t) \) and \( r^H \cdot X(rt) \) are statistically self-similar with respect to parameter \( H \)
- \( 0 \leq H \leq 1 \), \( H \) determines the roughness of the curve
- FBm is properly rescaled by dividing the amplitudes by \( r^H \)
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Fractal Brownian Motion (FBm)

- Effects of Hurst Exponent:
  - H = 0.5: Brownian motion, - increments are not correlated and independent
  - H > 0.5: Rather smooth curves, - increments have positive correlation
  - H < 0.5: Rough, erratic curves, - increments have negative correlation
- FBm with 0.5 < H < 0.9 looks very similar to contour lines of terrains

Fractal Brownian Motion (FBm)

- FBm is $1/f^\beta$-noise
- Fractal dimension $D = d + 1 - H = d + (3 - \beta)/2$
  - d indicates the topological dimension of $X(t)$
  - d=1: $X(t)$ - curves, $D = 2 - H = (5 - \beta)/2$
  - d=2: $X(s,t)$ - terrain, $D = 3 - H = (7 - \beta)/2$
  - d=3: $X(r,s,t)$ - clouds, $D = 4 - H = (9 - \beta)/2$
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Fractal Brownian Motion (FBm)

- H = 0.2
  - D = 1.8
- H = 0.5
  - D = 1.5
- H = 0.8
  - D = 1.2

Midpoint Displacement

- Used by Archimedes (287-212 BC) for parabola construction
  \[ P(x) = a - bx^2, \quad b > 0 \]
- Successive subdivision of a chord and displacement of the division points
- The midpoints of the k\(^{th}\) subdivision step are displaced by \(4^{-k\delta}\), where \(\delta = (x_A + x_B)/2\) and \(x_A, x_B\) are the endpoints of the subdivided chord
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Midpoint Displacement

- Construction of the parabola

\[ \frac{\delta}{8}, \frac{\delta}{16}, \frac{\delta}{4} \]

Takagi used the method (1900) with the displacement rule \(2^{-kd}\) which results in a self similar fractal curve.

Landsberg picked \(w\) arbitrary out of \((0.5, 1)\) and performed the method with \(w^k\), which also results in a self similar fractal curve.
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Midpoint Displacement

- Takagi curve
- Landsberg curve

Using random displacement values to approximate FBm

\[ X(t) \]

\[ D_1 \]

\[ D_2r \]

\[ D_{2l} \]
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**Midpoint Displacement**

- Step 1: \( X(1/2) = (X(0) + X(1))/2 + D_1 \)
- Step \( n \): Linear interpolation and displacement between two adjacent points
  
  - \( D_n \) is a normal distributed random variable
    
    \[
    E[D_n] = 0 \quad \text{Var}[D_n] = (1 - 2^{2H-2})/2^{2nH}
    \]
  - \( D_n = N(0, \delta_n) = \delta_n N(0, 1), \delta_n = \delta_{n-1} \cdot 0.5^{H/2} \)
  - \( X(t) \) and \( 1/2^hX(2t) \) are statistically self similar

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**Midpoint Displacement**

- 8 subdivision steps

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**Midpoint Displacement**

- Only a coarse approximation of FBm
- Not correlated
- Subdivided parts are calculated independently of each other
- Visible artefacts at subdivision points, can be avoided by displacing all points in step $k$ with $D_k$, $k=1,2,...,n$
- Efficient method, subdivision depth can easily be adapted to image resolution

**Generating Terrains**

- **Carpenter’s method**: Subdividing triangles:
  - Midpoint displacement is performed on the edges, $\Rightarrow$ 4 new triangles are generated
  - Continue until the triangles are small enough
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Generating Terrains

- Adjacent triangles have no influence on the subdivision of a triangle
- Visible artefacts along the edges (the mesh can be seen)

Better approach: Diamond & Square algorithm
Subdivision of a square is influenced by adjacent squares, ⇒ better approximation of FBm

old points  new points
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Generating Terrains

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7.22
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Generating Terrains

- Mandelbrot’s suggestion: Use random numbers with a skew distribution, \( \Rightarrow \) more realistic landscapes

Design of landscapes:
- The shape of a terrain is only determined by pseudo random numbers
- No way to predict the shape for any seed value for the random number generator
- Solution: Merge two height fields with an \( \alpha \)-channel

\[ F = \alpha \cdot F_1 + (1-\alpha) \cdot F_2 \]
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Zoom Sequence

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Generating Clouds

- Draw the height field as color map with different transparency:
  - If \( F[i][j] < \) threshold then set the pixel to full transparency
  - Else set the transparency according to \( F[i][j] \)
- Results in a 2D-model of clouds, ⇒ suitable for background rendering and environment mapping
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7.26

Generating Clouds

Ray Tracing of Terrains

- Visualization of a terrain demands much more effort than generating it
- Terrains are represented as fine triangle mesh, \(\Rightarrow\) huge number of triangles
- Ray-terrain intersection should be fast and memory consumption should be low
- Kajiya’s method (1983): Generate only those parts of the terrain which are necessary for intersection calculation
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Ray Tracing of Terrains

Kajiya’s method:
- Each triangle is embedded into a prism which encloses the part of the terrain that is generated out of the triangle.
- If a ray intersects a prism then midpoint displacement is applied to the enclosed triangle.
- Four new prisms are generated which enclose the four new triangles.

The height of each prism is estimated according to stochastic properties of FBm.

The intersected prisms are stored in a list and are sorted by their distance from the eye point.

The recursion depth can be adapted to different levels of detail.
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Ray Tracing of Terrains

Kajiya’s method:

- Low memory consumption, ⇒ high accuracy of approximation possible
- Efficient intersection calculation
- Complex shading models can be applied to fractal terrains

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Ray Tracing Terrain

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Ray Tracing Terrain

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7.29
### Spectral Synthesis Method

- **Approximation of FBm by generating a spectral distribution of the form** $1/f^\beta$
- **Inverse Fourier transform** is used to obtain $X(t)$ in the time domain

$$X(t) = \sum_{k=1}^{n/2} (A_k \cos kt + B_k \sin kt)$$

- **Good approximation of FBm without artefacts**
- **Summing different noise octaves**

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### Spectral Synthesis Method

- **High computation time**
- **Does not proceed in stages of increasing spatial resolution**
- **High detail demands the same computation as low detail**
- **The whole object must be generated (also if only a small part is needed)**
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Spectral Synthesis Method

- Adding higher frequencies

- Turbulence
  - Add absolute noise
  - Makes everything positive

Spectral Synthesis Method
7. Stochastic Fractals

Spectral Synthesis Method

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63

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64
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Spectral Synthesis Method

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65

66

7.33
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Spectral Synthesis Method

- Suitable for the generation of ocean waves
- Ocean waves are not approximated by FBm but by a spectral distribution of the form $1/f^5$, $\Rightarrow$ not fractal
- Clouds can be generated by modification of the volume density of ellipsoids according to 3D-FBm
- Visualization with advanced volume rendering techniques

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