4. Chaotic Behavior of Fractal Attractors

- Study the dynamic behavior of fractals
- Understand the stochastic method
- Assignment of an address to the points of an attractor $A_\infty$
- Examination of the address space with dynamic system theory
- Chaotic properties of the address space can be transferred to the attractor

Addresses of Fractal Attractors

- Given: IFS $\{X; f_1, f_2, \ldots, f_n\}$ with the attractor $A_\infty = f_1(A_\infty) \cup \ldots \cup f_n(A_\infty)$, decomposed in $n$ subsets
- Definition of the address of a point $x \in A_\infty$:
  - $x \in f_a(A_\infty) \Rightarrow$ address of $x$: $a \ldots$
  - $x \in f_af_b(A_\infty) \Rightarrow$ address of $x$: $ab \ldots$
- Infinite many indices are necessary to identify a point $x \in A_\infty$ exactly
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Addresses of Fractal Attractors

Hierarchy tree

\[ A_\infty \cup f_1(A_\infty) \cup f_2(A_\infty) \]

Address tree

\[ \sum_2 \]

\[ \begin{array}{c}
1 \ldots \\
11 \ldots \\
12 \ldots \\

2 \ldots \\
21 \ldots \\
22 \ldots \\
\end{array} \]

Subtriangles of the Sierpinski gasket and the corresponding addresses of \( \sum_3 \)
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**Addresses of Fractal Attractors**

- Example: Addresses of subtriangles & points

![Diagram showing addresses of fractal attractors](image)

**Address Space**

- Def.: The **metric address space** $(\Sigma_n, d_S)$ for an IFS $\{X; f_1, f_2, \ldots, f_n\}$ is defined by

$$\Sigma_n = \{\sigma_1 \sigma_2 \ldots \sigma_i \ldots \mid \sigma_i \in \{1 \ldots n\}\}$$

and the **Symbol Metric**

$$d_s(\alpha, \beta) = \sum_{i=1}^{\infty} \frac{|\alpha_i - \beta_i|}{(n+1)^i}$$

$\alpha = \alpha_1 \ldots \alpha_m \ldots$

$\beta = \beta_1 \ldots \beta_m \ldots$
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**Address Space**

- Def.: The mapping $\phi: \sum_n \rightarrow A_\infty$ is defined as
  $\phi(\sigma) = f_{\sigma_1}f_{\sigma_2} \ldots f_{\sigma_m} \ldots (x) = a \in A_\infty$,
  $\sigma \in \sum_n$, $x$ arbitrary

- $\phi(\sigma)$ is independent of the starting point $x$

- $\phi$ is a continuous mapping, similar addresses correspond to nearby points

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**Address Space**

- The metric spaces $(\sum_n, d_S)$ and $(A_\infty, d)$ have the same properties

- Thus the dynamic behaviour of $A_\infty$ can be analyzed by examining the dynamic behaviour of $\sum_n$

- $\phi^{-1}(a) = \{\sigma \in \sum_n: \phi(\sigma) = a\}$, $a \in A_\infty$, set of all addresses for a point $a$
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**Address Space**

- If each point of $A_\infty$ has a unique address, then $A_\infty$ is called totally disconnected.
- The Sierpinski gasket consists of infinite many branching (touching) points, thus only corner points have a unique address.

![Address Space Diagram]

**Dynamic Systems**

- Def.: Let $(X,d)$ be a metric space and $f: X \rightarrow X$ a function, then $\{X, f\}$ is called dynamic system.
- Def.: The sequence $\{f_n(x)\} = \{x, f(x), f^2(x), \ldots\}$ is called orbit of $x \in X$.
- Example: $(\sum_n, T)$ is a dynamic system, $T: \sum_n \rightarrow \sum_n$ is called shift function and defined as $T(\sigma_1 \sigma_2 \sigma_3 \sigma_4 \ldots) = \sigma_2 \sigma_3 \sigma_4 \ldots$.
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**Dynamic Systems**

- An IFS \( \{X; f_1, f_2, \ldots, f_n\} \) defines a special dynamic system \( \{H(X), W\} \), where \( W \) is the Hutchinson operator.

- The most interesting dynamic systems operate with non-linear functions (Julia sets, Mandelbrot set, strange attractors).

**Dynamic Systems**

- Orbit of \( \{R, f\} \): the fixpoints are the intersection points of the median with the graph.
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Fixpoints

- Def.: \( \{X,f\} \) is a dynamic system, a point \( p \in X \) is called **periodic point** if there exists a number \( n > 0 \), so that \( f^n(p) = p \). 
  
  \( \{p, f(p), \ldots, f^n(p)\} \) is called **cycle** of \( p \) and \( n \) is called **cycle length** or **period** of \( p \).

- \( p \) periodic in \( \{X,f\} \) \( \iff \) \( p \) is fixpoint of \( \{X,f^n\} \)

- Minimal period of \( p = \min\{n \mid f^n(p) = p\} \)

Fixpoints

- Def.: A fixpoint \( x=f(x) \) of \( \{X,f\} \) is called **attractive**, if there exists an \( \varepsilon > 0 \), so that \( f \) is a contraction mapping in \( B(x,\varepsilon) \).

- **Repelling**, if there exists an \( \varepsilon > 0 \), \( s > 1 \), so that \( d(x,f(y)) \geq s \cdot d(x,y) \), \( \forall \ y \in B(x,\varepsilon) \)

attractive point

repelling point

Christoph Traxler
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Fixpoints

- The dynamic system \( \{H(X), W\} \) of an IFS has exactly one fixpoint, no periodic points, all orbits converge to \( A_\infty \).
- Usually dynamic systems have several fixpoints.
- A periodic point \( p \) with period \( n \) is called attractive (repelling), if \( p \) is an attractive (repelling) fixpoint of \( \{X, f^n\} \).

Example: \( \{R, f\}, f(x) = \lambda x(1-x), \lambda < 3 \)
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**Fixpoints**

- Example: \( \{R, f\}, f(x) = \lambda x(1-x), \lambda > 3 \)

![Diagram of fixpoints](image)

**IFS Attractor as Dynamic Systems**

- Def.: \( \{X; f_1, f_2, \ldots, f_n\} \) is an IFS, the shift transformation \( S: A_\infty \rightarrow A_\infty \) is defined as
  \[ S(a) = f_i^{-1}(a), \quad \forall \ a \in f_i(A_\infty) \]
  \( \{A_\infty, S\} \) is a dynamic system

- \( f^m(A_\infty) \cap f^n(A_\infty) \neq \emptyset \Rightarrow S(a) \) is ambiguous
  \[ \forall \ a \in f^m(A_\infty) \cap f^n(A_\infty) \]

- Orbits of points from \( A_\infty \) can be examined in \( \{A_\infty, S\} \) (backward orbits)
4. Chaotic Behavior of Fractal Attractors

**IFS Attractor as Dynamic Systems**

- Homomorphism between \( \{\sum_n, T\} \) and \( \{A_\infty, S\} \)
  - Function \( T \) has the same effect in \( \sum_n \) as \( S \) in \( A_\infty \)
  - The relation is established by \( \phi: \sum_n \rightarrow A_\infty \)
    \[ \phi(T(\sigma)) = S(\phi(\sigma)) \]

**Diagram:***

\[ \sum_n \xrightarrow{T} \sum_n \xrightarrow{\phi} A_\infty \]
\[ \sum_n \xrightarrow{S} A_\infty \]

**IFS Attractor as Dynamic Systems**

- The knowledge of the dynamic behavior of \( \{\sum_n, T\} \) can be applied to \( \{A_\infty, S\} \) as well
- It is much easier to examine the properties of \( \{\sum_n, T\} \) than of \( \{A_\infty, S\} \)
- \( \{\sum_n, T\} \) contains no geometric but only topological information about \( \{A_\infty, S\} \)
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Orbits of IFS Attractors

Example:
\[
\{\sum_{n=1}^{\infty} T_n \} \xrightarrow{\phi^{-1}} \{A_\infty, S\}
\]

\[
\begin{align*}
&a_0 \\
&1231322 = a_0 \\
&231322 = a_1 = f_1^{-1}(a_0) \\
&31322 = a_2 = f_2^{-1}(a_1) \\
&1322 = a_3 = f_3^{-1}(a_2) \\
&322 = a_4 = f_1^{-1}(a_3) \\
&22 = a_5 = f_3^{-1}(a_4) \\
&22 = a_6 = f_2^{-1}(a_5) = a_5
\end{align*}
\]

Examples:

Cantor set, orbit of the point \[ s = 123111213212 \]

Orbit in Barnsely’s fern
4. Chaotic Behavior of Fractal Attractors

Chaos in Dynamic Systems

Def.: A dynamic system \( \{X, f\} \) is called chaotic if:

1. \( \{X, f\} \) is transitive
2. \( \{X, f\} \) is sensitive with respect to the starting conditions
3. The set of periodic orbits of \( f \) is dense in \( X \)

Orbits of points taken from an arbitrary small subset reach every part of \( X \)

Def.: \( \{X, f\} \) is transitive, if there exists a number \( n \) for the open sets \( U, V \subseteq X \) so that \( U \cap f^n(V) \neq \emptyset \)
4. Chaotic Behavior of Fractal Attractors

Chaos in Dynamic Systems

- \( \{\Sigma^n, T\} \) is transitive:
  - Any open set \( V \) with \( \forall \omega \in V \) exists a \( B(\sigma, \varepsilon) = \{\omega: d(\sigma, \omega) < \varepsilon\} \), \( B(\sigma, \varepsilon) \subseteq V \)
  - Example \( n=2 \):
    \[
    \sigma = \sigma_1 \sigma_2 \ldots \sigma_m \ldots \\
    \omega_1 = \sigma_1 \sigma_2 \ldots \sigma_m 11 \ldots \\
    \omega_2 = \sigma_1 \sigma_2 \ldots \sigma_m 12 \ldots \\
    \omega_3 = \sigma_1 \sigma_2 \ldots \sigma_m 21 \ldots \\
    \omega_4 = \sigma_1 \sigma_2 \ldots \sigma_m 22 \ldots 
    \]
  - The \( \omega_i \) can be continued with any possible combination of symbols

Chaos in Dynamic Systems

- \( \{\Sigma^n, T\} \) is transitive:
  - \( \omega \in B(\sigma, \varepsilon) \Rightarrow \omega = \sigma_1 \sigma_2 \ldots \sigma_m \omega_1 \omega_2 \ldots, \omega_i \in \{1, \ldots, n\} \)
  - \( \Rightarrow T^m (\omega) = \omega_1 \omega_2 \ldots \)
  - \( \Rightarrow T^m (B(\sigma, \varepsilon)) = \Sigma \)
  - \( \Rightarrow T^m (V) \cap U \neq \emptyset, \forall U, V \subseteq X \)
4. Chaotic Behavior of Fractal Attractors

Chaos in Dynamic Systems

- **Def.:** \( \{X,f\} \) is sensitive towards starting conditions if there exists a number \( d > 0 \), so that \( \forall x \in X \) and \( \forall B(x,\varepsilon), \varepsilon > 0, \exists y \in B(x,\varepsilon) \) and \( n > 0 \), so that \( d(f^n(x), f^n(y)) > d \)

- Nearby orbits move away from each other

\[
\begin{align*}
B(x,\varepsilon) & \quad \circ \quad f^n(x) \\
y & \quad \circ \quad f^n(y)
\end{align*}
\]

Chaos in Dynamic Systems

- \( \{\Sigma_n,T\} \) is sensitive:
  - \( \mu = \sigma_1 \sigma_2 \ldots \sigma_m 1 \ldots \quad T^m(\mu) = 1 \ldots \)
  - \( \nu = \sigma_1 \sigma_2 \ldots \sigma_m 2 \ldots \quad T^m(\nu) = 2 \ldots \)
  - \( d_S(\mu,\nu) = 1/(n+1)^m \)
  - \( d_S(T^m(\mu),T^m(\nu)) = 1/(n+1) \)
  - \( \mu \) and \( \nu \) are the starting conditions for \( T \)
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Chaos in Dynamic Systems

- Def.: \((X, d)\) is a metric space, a set \(S \subseteq X\) is called dense in \(X\), if for each point \(x \in X\) there exists a sequence \(\{s_n\}\) in \(S\), with \(\lim s_n = x\) 
  \((X\) is called closure of \(S)\)
- Example: The set of rational numbers \(Q\) is dense in \(R\)

Chaos in Dynamic Systems

- The set of periodic points \(P\) of \(\{\Sigma_n, T\}\) is dense in \(\Sigma_n\):
  - Arbitrary address \(\sigma\) & sequence of periodic addresses
    \[\sigma_1 \sigma_2 \ldots \sigma_m \ldots\]
    \[s_1 = \overline{\sigma_1}\]
    \[s_2 = \overline{\sigma_1 \sigma_2}\]
    \[\vdots\]
    \[s_m = \overline{\sigma_1 \sigma_2 \ldots \sigma_m}\]
4. Chaotic Behavior of Fractal Attractors

Chaos in Dynamic Systems

Fixpoints of $\Sigma_n$ are repelling points:

Example $n = 2$:

- $\mu = \overline{1}$, fixpoint
- $\nu = 1112 \ldots$, Arbitrary
- $d_S(\mu, \nu) = 1/34$, $d_S(T^3(\mu), T^3(\nu)) \approx 1/3$
- Periodic points of $\Sigma_n$ are repelling points
- Periodic points are dense in $\Sigma_n \Rightarrow \Sigma_n$ is densely covered by repelling points

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Chaos in Dynamic Systems

$\{\Sigma_n, T\}$ is a chaotic dynamic system, because:

(1) It is transitive

(2) It is sensitive towards starting conditions

(3) Set of periodic orbits of $T$ is dense in $\Sigma_n$

$\{\Sigma_n, T\}$ chaotic $\xrightarrow{\phi} \{A_{\infty}, S\}$ chaotic
Back to the Stochastic Method

1. The attractor \( A_\infty \) of an IFS \( \{X; f_1, f_2, \ldots, f_n\} \) is approximated by a random sequence of points \( x_n = f_i(x_{n-1}), i \in \{1, \ldots, n\} \)

2. Example: \( n=2 \), Random sequence 112 ...1:
   - Point \( x_0 \), \( x_1 = f_1(x_0) \), \( x_2 = f_1(x_1) \), \( x_3 = f_2(x_2) \), \ldots, \( x_{\text{max}} \)
   - Address \( \sigma \), \( 1\sigma \), \( 11\sigma \), \( 211\sigma \), \ldots, \( \sigma_{\text{max}} \)

3. The sequence \( \{x_{\text{max}}, \ldots, x_3, x_2, x_1, x_0\} \) is an orbit of the chaotic system \( \{A_\infty, S\} \)

Back to the Stochastic Method

1. Dynamic system \( \{\Sigma_n, T\} \), starting point \( \sigma \):
   - \( \sigma \) periodic, quasiperiodic:
     - Orbit \( \{T^n(\sigma)\} \) converges to periodic repelling point
   - \( \sigma \) not periodic:
     - Orbit \( \{T^n(\sigma)\} \) comes closer to a periodic repelling point from which it moves away
     - New points of \( A_\infty \) are always generated
4. Chaotic Behavior of Fractal Attractors

Back to the Stochastic Method

- How frequent are starting points $\sigma$ with a too short period?
- $U(p)$ - number of periodic orbits with minimal period $p$

$$U(p) = (n^p - \sum_{k \text{ divides } p} k \cdot U(k))/p$$

Example $\{\Sigma_2, \Sigma\}$:

- $U(1) = 2$  fixpoints 1, 2
- $U(2) = 1$  fixpoint 12
- $U(3) = 2$
- $U(10) = 99$
- $U(15) = 2182$
- $U(20) = 52377$
- $U(p)$ is a fast increasing function
4. Chaotic Behavior of Fractal Attractors

Back to the Stochastic Method

- If the starting point $\sigma$ is periodic, then its period is very long with high probability.
- Orbits of chaotic dynamic systems are distributed among the whole attractor.
  - $\{x, f_1(x), f_2 f_1(x), f_2 f_1 f_1(x), ...\} x \in A_\infty$, covers $A_\infty$.
  - $\{p, f_1(p), f_2 f_1(p), f_2 f_1 f_1(p), ...\} p \notin A_\infty$, converges to $\{x, f_1(x), f_2 f_1(x), f_2 f_1 f_1(x), ...\}$

Conclusion

- Examination of the chaotic behavior of fractal attractors of an IFS $\{X; f_1, f_2, ..., f_n\}$.
- Introduction of the address space $\Sigma_n$ and the dynamic system $\{\Sigma_n, T\}$.
- Relation between $\{\Sigma_n, T\}$ and $\{A_\infty, S\} \rightarrow$ chaotic properties of $\{\Sigma_n, T\}$ can be transferred to $\{A_\infty, S\}$.
4. Chaotic Behavior of Fractal Attractors

Conclusion

- The stochastic method can be analyzed with (backward) orbits in \( \{A_\infty, S\} \)
- \( \{A_\infty, S\} \) is chaotic \( \Rightarrow \) orbit is distributed among the whole attractor with very high probability
- The random orbit generates a good approximation of \( A_\infty \)