CRITICISM OF HUNTING MINIMUM WEIGHT TRIANGULATION EDGES

Andrej Ferko, Ľudovít Niepel, Tomáš Plachetka

Department of Computer Graphics and Image Processing
Comenius University
Bratislava, Slovakia

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Abstract. Minimum Weight Triangulation problem (MWT) is to find a set of edges of minimum total length that triangulates a given set of points in the plane. Although some properties of MWT have been proved and many heuristics proposed, polynomiality (and)/or NP-completness of MWT problem is still unsolved. The problem belongs to the few open problems from the book [GaJo]. In this paper we present results indicating that even very good approaches based on the local edge examination (like LMT-skeleton) fail to come close to the MWT for specially constructed set of points. Furthermore, we show a method how to construct a set of points for which MWT is unstable — a slight displacement of a point in the input set causes significant change in the price of MWT.

1. Local MWT edge examination

Up to now it is not known yet if there is a polynomial algorithm which finds the MWT for an arbitrary point set. Known polynomial algorithms related to MWT problem can be divided into two groups:

1. algorithms approximating the MWT by a triangulation which may not be the minimum one;
2. algorithms which attempt to find a maximum subgraph of MWT.

1.1. Algorithms approximating the MWT.

Algorithms belonging to the first group are usually heuristics based on observing some features of the MWT. Here is a list of several such algorithms:

- Delaunay triangulation;
- greedy algorithm;
- Plaisted's and Hong's heuristics [PlHo];
- simulated annealing [BFMNPS];
- pairwise edge acceptation [BFMNPS];
- etc.

However, all the algorithms mentioned above compute only an approximation of MWT (a special case is simulated annealing, which is a probabilistic algorithm).

1.2. Algorithms constructing a subgraph of MWT.

Algorithms from the second group are based on theorems characterizing edges which must belong to MWT. In combination with heuristics from the first group they can achieve very good practical results (computing as many MWT edges as possible and applying a good heuristic on the rest of edges). The most powerful algorithm was proposed by Dickerson and Montague in [DiMo]. It is based on constructing the LMT-skeleton (Locally Minimum Triangulation) which is a subgraph of MWT (LMT-skeleton can be computed in $O(n^4)$ time, where n is the size of input point set). On randomly generated input sets was experimentally shown that this subgraph is usually quite large, so the exact MWT can be computed for several hundred points. (Edges of the MWT which do not belong to the LMT-skeleton are computed by applying brute force algorithm on each discontinuity component of LMT-skeleton.)

Let us concentrate on known subgraphs of the MWT.

2. Survey of polynomial time detection of MWT edges

In this section we summarize known algorithms producing a subgraph of the exact MWT in polynomial time and consider limitations of some of them.

Up to now, the following edges of the MWT can be computed in polynomial time:

1. "Trivial edges" — those which are not intersected by any other edge. The convex hull of the input point set is included in this category.
2. Double edges of the nearest neighbour graph (NNG)". This result was presented in [YaXuYo]. (We shall show that this result can be slightly improved.)
3. Edges of 1.17682 β-skeleton. [Ke], [ChXu]
4. Edges of the LMT-skeleton. [DiMo]
5. The shortest edge. [Gi]

Below we describe these categories in more detail.

2.1. Double edges of the nearest neighbour graph (NNG).

In [YaXuYo] was proved that all double edges of the NNG must belong to the MWT. We shall prove that the result can be improved a little by weakening assumptions of the theorem:

Theorem. *Let X, Y be points of the input point set $\mathcal{S}$, such that intersection of circles $circle(X, |XY|), circle(Y, |XY|)$ and area defined by perpendicular displacement of line segment $\overline{AB}$ does not contain any other point of $\mathcal{S}$. Then the edge AB belongs to the MWT of point set $\mathcal{S}$.*

Proof.

2.1.1. Overview.

If the assumptions of the theorem are satisfied and $\overline{AB}$ is a line segment intersecting $\overline{XY}$, the following observations hold:

Observation 1. $|AB| > |XY|$.

Observation 2. $\measuredangle AXB > \pi/2$ and $\measuredangle AYB > \pi/2$.

Observation 3. $|AX| < |AB|$ and $|BX| < |AB|$.

Let $\mathbb{S}$ be a set of points in the plane and $\mathbb{M}$ the minimum triangulation of $\mathbb{S}$. Let us assume that an edge XY satisfies assumptions of the theorem and $XY \notin \mathbb{M}$. Then (from definition of triangulation) it must be intersected by at least one edge. From those *we choose the edge AB with minimum length*. We will show that in that case edges XA, XB, YA, YB also belong to $\mathbb{M}$ and quadrangle $XAYB$ contains no point of $\mathbb{S}$. By flipping edges XY, AB we obtain a triangulation with less total length. Thus we get a contradiction with minimality of $\mathbb{M}$.

To prove the contradiction above, we will separately consider halfplanes determined by line AB, situation in both halfplanes being similar — let us concentrate on halfplane $X\overleftrightarrow{AB}$. First of all, we shall show that the triangle XAB does not contain any point of $\mathbb{S}$. Then there are two possible cases:

Case A: One of the edges XA, XB belongs to $\mathbb{M}$ (and the other one does not).
Case B: None of the edges XA, XB belongs to $\mathbb{M}$.

We shall show that both cases lead to a contradiction with our assumptions.

2.1.2. Triangle XAB does not contain any point of $\mathbb{S}$.

Let the triangle XAB contain a point P which lies, say, in halfplane $A\overleftrightarrow{XY}$. Then length $|PB|$ is certainly less than length $|AB|$. Edge PB cannot be in $\mathbb{M}$, because AB is the shortest edge crossing XY. Thus PB must be crossed by some edge P_1P_2 where at least one point P_1 or P_2 belongs to the triangle XAB. We can repeat our argument for the point closer to AB. Fact that there is only finite number of points in $\mathbb{S}$ leads to a contradiction with minimality of AB.

2.1.3. Case A: One of the edges XA, XB belongs to $\mathbb{M}$ (and the other does not).

Let the edge XB be in $\mathbb{M}$. $AX \notin \mathbb{M}$ implies that some edge $PB \in \mathbb{M}$ (the edge must cross AX and we already proved that the triangle AXB contains no point of $\mathbb{S}$). We can select the point P so that length of PB is minimum of all edges crossing XA.

If both edges XP, AP were in $\mathbb{M}$, we could replace the edge PB with AX, thus getting a shorter triangulation. If at least one of XP, AP is not in $\mathbb{M}$, there must exist points $P_1, \ldots, P_n \in \mathbb{S}$ such that $\{P_1B, \ldots, P_nB, PB\}$ is set of all edges of $\mathbb{M}$ which intersect either XP or AP (there is no point in triangle APX, because PB is the shortest edge crossing XA). Let $\mathbb{I} = \{XB, P_1B, \ldots, P_nB, PB, AB\}$. The convex hull of the triangles between BX and BA form a natural ordering of points P_i. It is easy to see that in the ordering we can always choose three consecutive edges $Z_1B, Z_2B, Z_3B \in \mathbb{I}$ so that the edge Z_1Z_3 is crossed only by edge Z_2B of $\mathbb{M}$.

But this is a contradiction with minimality of $\mathbb{M}$, because Z_1Z_3 is certainly shorter than Z_2B ($\measuredangle Z_1BZ_3 < \pi/2$) and so we can replace the longer edge with the shorter one. Thus $XB \in \mathbb{M}$ implies $XA \in \mathbb{M}$.

2.1.4. Case B: None of the edges XA, XB belongs to $\mathbb{M}$.

We showed that the triangle XAB does not contain any point of $\mathbb{S}$. This implies that if an edge $EF \in \mathbb{M}, E \neq A, E \neq B, F \neq A, F \neq B$ crosses one of edges XA, XB, then e also crosses the second one.

Let X' be the intersection point of XY and AB. Let us consider all edges crossing line segment $\overline{XX'}$ — let $\mathbb{C}$ denote the set after adding edge AB. The edges define a set of points $\{A_1, \ldots, A_m\}$ above $\overline{XX'}$ and $\{B_1, \ldots, B_n\}$ under $\overline{XX'}$ (points in the sets are ordered by their x-coordinate). Basic observations:

a) Each of the points coincides with at least two edges of $\mathbb{C}$.

b) $e \in \mathbb{C} \Longrightarrow |e| \geq |AB|$ (we assume minimality of AB among edges crossing XY).

c) If $A_iB_j \in \mathbb{C}$, then $|XA_i| < |A_iB_j|$ and $|XB_j| < |A_iB_j|$. Furthermore, $|XA| < |AB|$ and $|XB| < |AB|$ (see Preliminaries).

d) If an edge $e \in \mathbb{M}$ is inside the (simple) polygon $XA_1 \ldots A_nABB_n \ldots B_1$, then $e \in \mathbb{C}$.

e) Edges which form the polygon mentioned in previous point are in $\mathbb{M}$ (otherwise there would exist an edge e crossing line segment $\overline{XX'}$, $e \notin \mathbb{C}$).

First we shall consider situation when all edges XA, $XA_1, \ldots, XA_m$, XB, $XB_1, \ldots, XB_n$ lie *inside* the polygon $XA_1 \ldots A_nABB_n \ldots B_1$ (in other words, all points A, $A_1, \ldots, A_m$, B, $B_1, \ldots, B_n$ are visible from X). Let us replace all triangulation edges from inside the polygon with the following ones: XA, XB, $XA_1, \ldots, XA_m$, $XB_1, \ldots, XB_n$. From the observations a) and c) follows that the length of the new triangulation will be less than the original one. This is a contradiction with minimality of $\mathbb{M}$.

In the second situation at least one of points A, $A_1, \ldots, A_m$, B, $B_1, \ldots, B_n$ is not visible from X. Let us assume that a point $Z \in \{A, A_1, \ldots, A_m\}$ is the case. To prove the contradiction with minimality of $\mathbb{M}$ we shall use a reasoning similar to the one from the section "3.Case A" — there must exist a convex polygon defined by three points of $\{A, A_1, \ldots, A_m\}$ and one point of $\{B, B_1, \ldots, B_n\}$. Within this polygon we can repeat the reasoning used in the previous section to obtain a contradiction with minimality of $\mathbb{M}$.

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2.2. Edges of the 1.17682 β-skeleton.

The 1.17682 β-skeleton is defined as set of all edges X, Y satisfying the following condition:

> Let Z_1, Z_2 be intersections of two circles with centers on line XY and radii $\beta = 1.17682 \cdot |XY|/2$, the circles passing through the points X, Y respectively. If the closed area defined by the circles does not contain any point of $\mathcal{S}$, then the edge XY belongs to the 1.17682 β-skeleton.

In [ChXu] was proved that all edges of the 1.17682 β-skeleton form belong to the MWT. It is obvious that the 1.17682 β-skeleton can be computed in polynomial time.

2.3. Edges of the LMT-skeleton.

The LMT-skeleton and its modifications are discussed in [DiMo]. It is based on the idea of "locally minimum triangulation":

> Let XY be an edge. If in every empty quadrilateral $AXBY$, where A lies to the left and B to the right of the line XY, holds $|AB| > |XY|$, then the edge XY belongs to the LMT-skeleton.

In [DiMo] was proved that the LMT-skeleton forms a subgraph of the MWT. An obvious upper bound for the LMT-skeleton computation is $O(n^4)$.

3. Weakness of the LMT-skeleton

The LMT-skeleton proved to produce almost complete subgraphs of the MWT on a large number of randomly generated point sets. Time complexity of the algorithm shows that the MWT can be constructed using the LMT-skeleton in time $O(n^{k+2})$, where k is number of unconnected components of the LMT-skeleton. Experiments with the LMT-skeleton on random point sets gave quite an optimistic result [DiMo]:

> *"Though we suspect an example exists, we have not yet found a point set where the LMT-skeleton contains more than a single unconnected region interior to a simple polygon. Thus even when the LMT-skeleton is not completely connected it should still allow the complete MWT to be constructed quickly.*
>
> ...
>
> *An open problem: What is the worst case disconnectivity of the LMT-skeleton?"*

We shall show now that the answers to the questions above are pessimistic — for a given n there always exists a special arrangement of n points such that number of disconnectivity components of LMT-skeleton is $\Theta(n)$. This implies that the time complexity of the proposed algorithm is asymptotically exponential in the worst case.

Observation. *For an input point set consisting of nine points (or more) equally distributed on a circle and one point in the center of the circle the LMT skeleton will not connect the central point.*

Let us construct a special arrangement of n points (n being a large number) for which the number of disconnectivity components is $\Theta(n)$. Let us begin with $\lceil n/10 \rceil$ circles with the same radii r, the distance of their centers being greater than $2r$ for any pair of the circles. We equally distribute 10 points on each circle as is described in the observation (one circle may differ in the number of points). As the LMT-skeleton will not connect the central points in the circles, the number of circles is less than or equal to the number of disconnectivity components — and this number is $\Theta(n)$.

4. Unstability of MWT

4.1. Motivation.
In section 2.1. we proved that if two points of the input set are the nearest neigbours of each other, then the edge connecting them must belong to the MWT. A question arises:

> *If we weaken assumptions of the theorem and allow a third point Z in the "forbidden" area of points X, Y, assuming edges ZX, ZY to be in MWT, will the edge XY still always be in MWT?*

In the case of positive answer, we would be able to detect additional MWT edges in polynomial time. However, in the next section we shall show that the answer is negative.

Let us consider an input point set like this: Let edges ZX, ZY be in MWT, point Z lie on the axis of the line segment $\overline{XY}$, Z being "very close" to $\overline{XY}$. Let two points A and B lie on the other side of $\overline{XY}$ "very close" to each other and "very close" to the axis of $\overline{XY}$, and so that polygon $ABYZX$ is convex. Let us denote $a = |XY|/2$, $b = |XA| \doteq |XB| \doteq |YA| \doteq |YB|$ (see Fig. 1). The MWT can appear in two forms, depending on the proportion b/a:

Case 1: convex hull+the edge XY and one of AX, AY, BX, BY;
Case 2: convex hull+both edges AZ, BZ.

The two situations are distinguished by unequalities $b + 2a \leq 2\sqrt{b^2 - a^2}$ (in Case 1), and $b + 2a \geq 2\sqrt{b^2 - a^2}$ (in Case 2). After some simplifications we get that $b/a \geq (2 + \sqrt{28})/3$ in Case 1, and $b/a \leq (2 + \sqrt{28})/3$ in Case 2.

Resume: for the special input point set described above we found a relationship between distances a and b, saying that the edge XY is in MWT iff the points A, B fulfil the required minimum distance from the points X and Y. The required minimum distance $(2 + \sqrt{28})/3$ is constant for given points X, Y, Z for the input point set described above. In the next section we show that there is no constant lower bound for the required minimum distance in general case (for given points XYZ).

4.2. A slight change of input point set causes an arbitrarily great weight change of MWT.

Let us make similar assumptions about points X, Y, Z as we did in the previous section, but consider K points $A_1, \ldots, A_K$ instead of A, B (points $A_1, \ldots, A_K$ have similar properties as A, B from the previous section and lie on the same line parallel to XY). The case when the edge XY belongs to MWT is characterized by the equation $2a + (K - 1)b \geq K\sqrt{b^2 - a^2}$. After some simplifications we get $b/a \geq 1 + \frac{-1+K\sqrt{2K+3}}{2K-1}$. For $K \to \infty$ the unequality can be rewritten as $b/a \geq \sqrt{K/2}$. Thus we see that we can always raise the required minimum distance of the points $A_1, \ldots, A_K$ by adding another point A_{K+1} to the input point set.

Furthermore, note that for a large K we can place the points $A_1, \ldots, A_K$ on a line parallel to XY so that if we shift them a little bit closer to XY, we get the other form of MWT. Let us shape the points A_i to the form of "$\bigcup$" (please imagine a bottom part of a w-i-d-e parabola, so that all the points A_i can "see" both points X, Y) above the edge XY and shift them the bit closer to XY (we get MWT-concave). All the points A_i will connect to the point Z. Now shape them to the form of "$\bigcap$" (again, imagine an upper part of a w-i-d-e parabola) and shift them the bit further from XY (we get MWT-convex). The weight of MWT-convex is much more less than the weight of MWT-concave (see Fig. 2 and Fig. 3). The proportionality factor $\frac{\text{weight(MWT-concave)}}{\text{weight(MWT-convex)}}$ grows with the number of points A_i, thus we can make the weight change arbitrarily great.

Figures

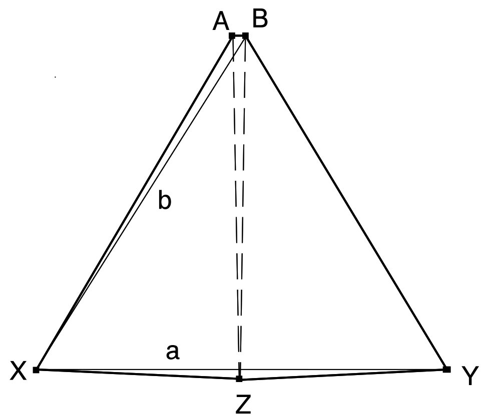

Fig. 1: The two forms of MWT

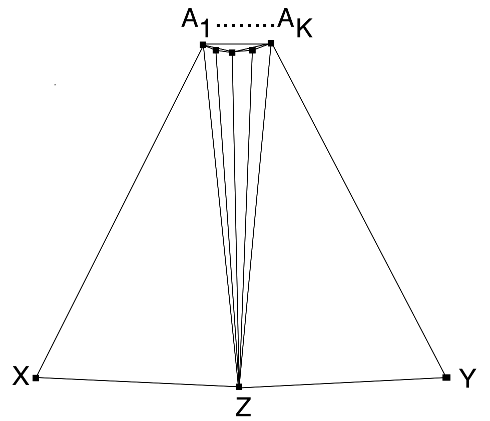

Fig. 2: MWT-concave

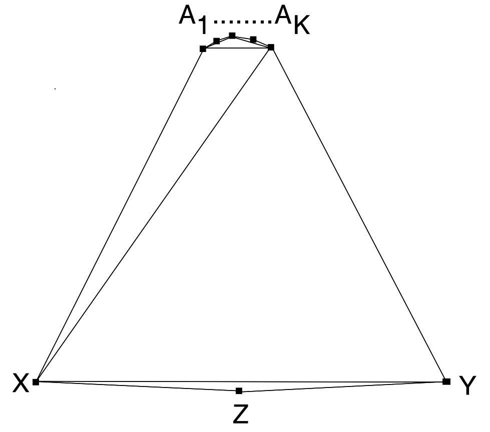

Fig. 3: MWT-convex

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Department of Computer Graphics and Image Processing, Comenius University, Mlynská dolina, 84215 Bratislava, Slovak Republic

E-mail: ferko@fmph.uniba.sk, niepel@fmph.uniba.sk, plachetka@fmph.uniba.sk