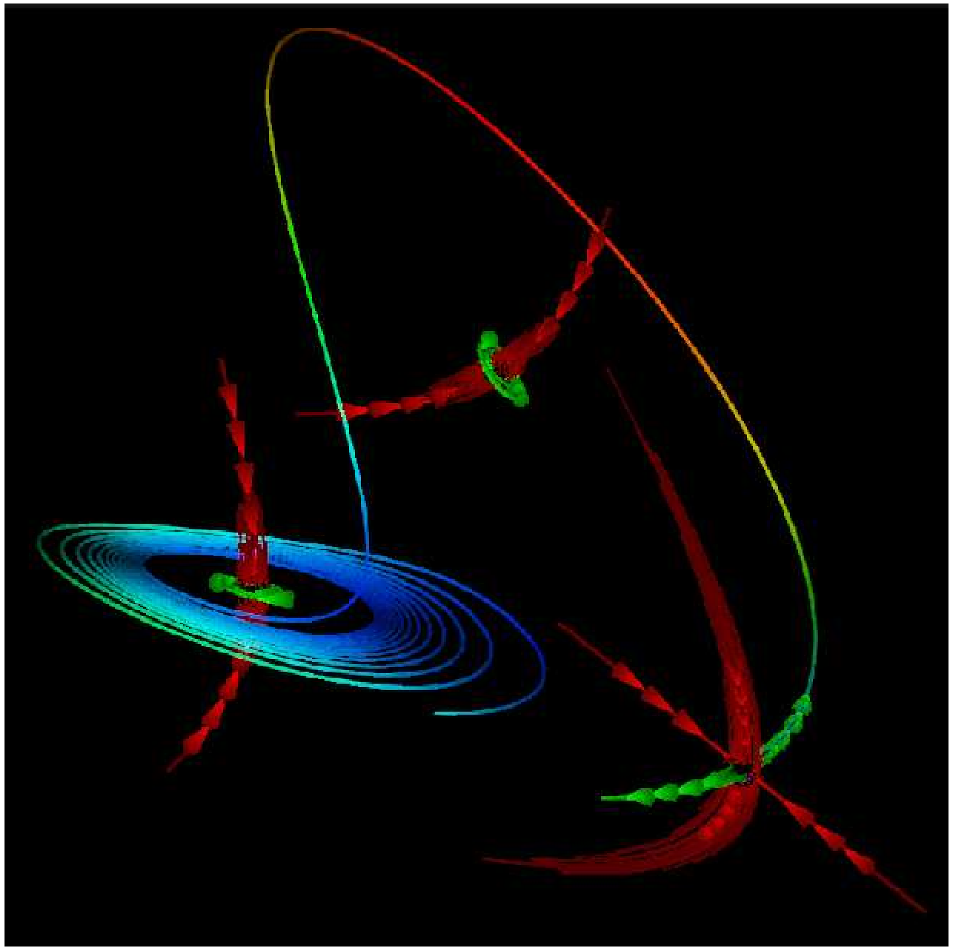
Visualizing complex dynamical systems in 3D



Helwig (still ;-) **Löffelmann**

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Overview

- **Introduction –** dynamical systems, visualization

- **More about dynamical systems –** some sorting
- **The Lotka-Volterra model –** an example of a dynamical system
- **Intuitive flow visualization –** stream lines, stream surfaces
- **Stream arrows –** enhanced direct flow visualization

- **Approaching a dynamical system –** some math
- **Meeting Lotka & Volterra again –** continue with extended example
- **Equilibria and visualization –** math and direct visualization combined!

- **Math continued towards complexity –** cycles, tori, chaotic attractors, etc.
- **Cycles and visualization –** facilitating **Poincaré sections** for visualization

- **Conclusions –** when **math meets visualization**, **maximal insight** is gained

Introduction

- **Dynamical systems –**
  Mathematical description of **temporary evolution**, **various application fields**

  Examples: predator-prey models (biology), models of chemical reactions (chemistry), social interactions (sociology), ...

- **Visualization –** exploiting the visual channel for ...
  Enabling **insight**, communicating **data**, support of **analysis & research**

  Examples: medical data vis. (scientific visualization), vis. of weather data (visualization for presentation), business data vis. (information visualization), ...

... following: more about dynamical systems ...

Dynamical systems - some sorting (next: example)

- **Analytical "brother"** of **vector fields –** **mathematical description** $\mathrm{f}()$ vs. **set of vectorical samples** (on a grid)

- **Discrete steps** or **continuous evolution –** **difference equations** or **differential equations** (usually ODEs)

$$\begin{array}{rcccl} \triangle\mathbf{x}(t) & = & \mathbf{x}(t+1) - \mathbf{x}(t) & = & \mathbf{f_p}(\mathbf{x}(t), t) \qquad \text{(discrete case, } t \in \mathbf{Z}\text{)} \\ \dot{\mathbf{x}}(t) & = & \mathrm{d}\mathbf{x}(t)/\mathrm{d}t & = & \mathbf{f_p}(\mathbf{x}(t), t) \quad \text{(continuous case, } t \in \mathbf{R}\text{)} \end{array}$$

- **Variables** vs. **parameters –** entities varying over time (usually called $\mathbf{x}$), parameters of the model itself ($\mathbf{p}$)

- **Time-dependent** or **steady –** Does $\mathrm{f}()$ change over time (like a river) or not (steady, time-independent system)

- **Dimension** of **phase space –** variables build up an n-dimensional Cartesian space

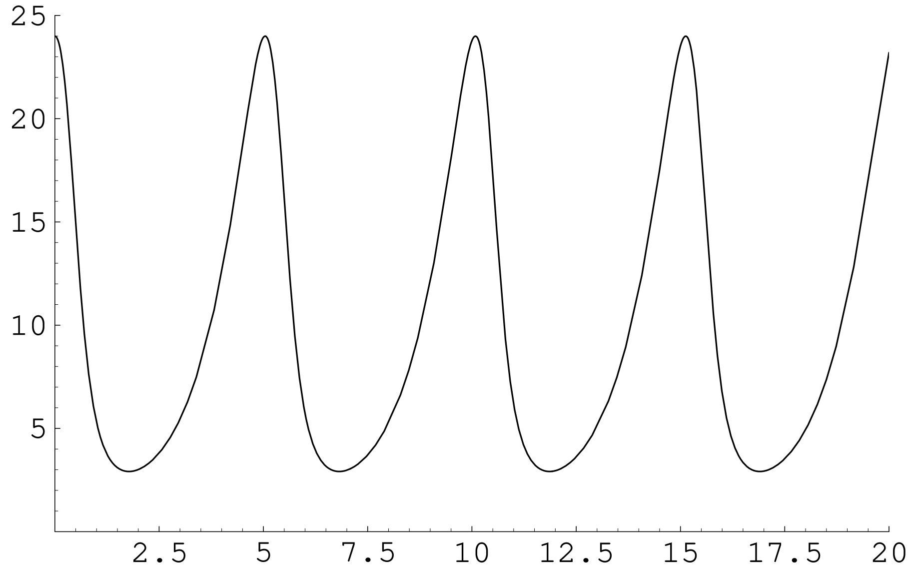
The predator-prey model by Lotka & Volterra

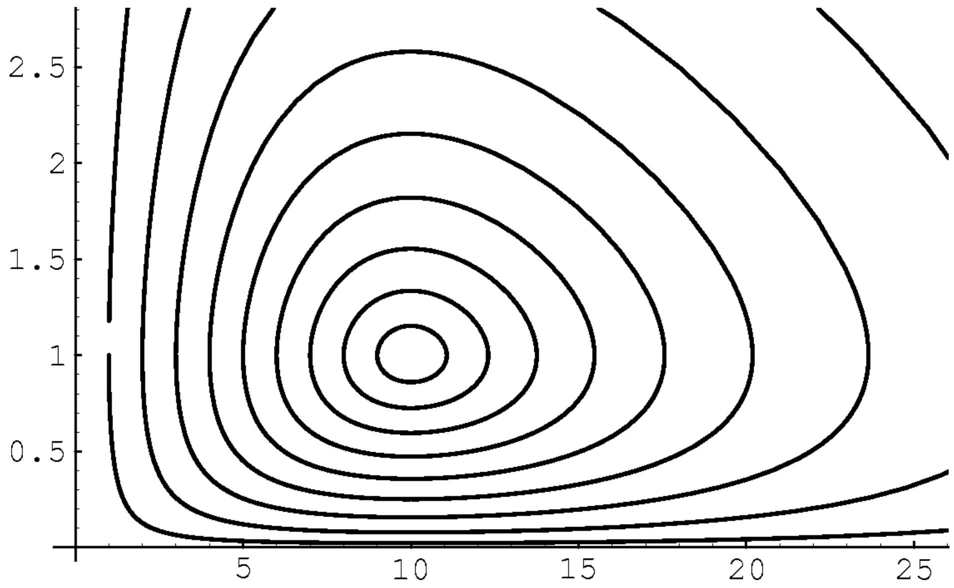
- **Two variables (prey x, predators y), four parameters (r, p, e, and m) –** prey reproduces exponentially (rate r), predators hunt prey (parameter p) and "use" prey for living (efficiency e), predators die (mortality m).

$$
\begin{aligned}
\dot{x} &= r \cdot x - p \cdot x \cdot y && \text{(evolution of prey)} \\
\dot{y} &= \qquad e \cdot p \cdot x \cdot y - m \cdot y && \text{(evolution of predators)}
\end{aligned}
$$

- **2-dimensional phase space, steady dynamical system (continuous) –** quite simple model, evolution is cyclic:
  −− little prey $\rightarrow$ number of predators decreases ($e \cdot p \cdot x \cdot y < m \cdot y$),
  +− less predators $\rightarrow$ prey regenerates again ($r \cdot x > p \cdot x \cdot y$),
  ++ more prey again $\rightarrow$ predators recover from starving ($e \cdot p \cdot x \cdot y > m \cdot y$),
  −+ many predators $\rightarrow$ good times of prey are over ($r \cdot x < p \cdot x \cdot y$),
  aso.

Lotka-Volterra: time-series, trajectories



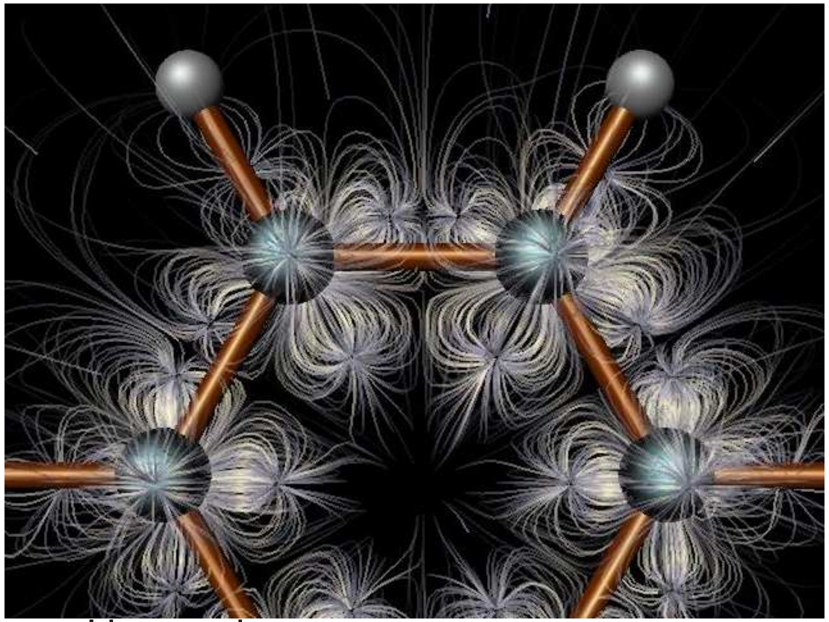


- **Time-series visualization** vs. **phase space plots –** 2D exploited for ...
  a plot of one variable (x) against time t or **trajectories** $\mathcal{T}_{\mathbf{s}}(t)$ in phase space (x, y)

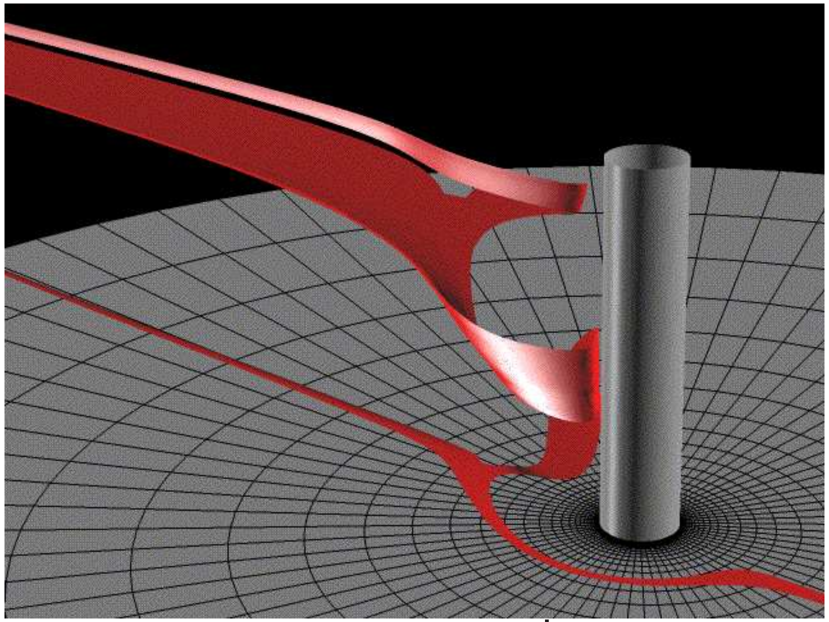
$$\mathcal{T}_{\mathbf{s}}(t) = \mathbf{s} + \int_{u=0}^{t} \mathbf{f}_{\mathbf{p}}(\mathcal{T}_{\mathbf{s}}(u))\,\mathrm{d}u$$

Stream lines, stream surface: direct visualization

- **Idea – select initial condition**, follow flow $\rightarrow$ numerical integration
- **(Explicit) Numerical integration (Euler, Runge-Kutta, etc.) –** iterate:
  (1) evaluate flow at current position: $\mathrm{f}(\mathrm{x})$, maybe look ahead: $\mathrm{x} + \Delta t \cdot \mathrm{f}(\mathrm{x})$, etc.
  (2) make tiny step along $\mathrm{f}(\mathrm{x})$ or $\mathrm{f}(\mathrm{x}) + \mathrm{f}(\mathrm{x}+\Delta t \cdot \mathrm{f}(\mathrm{x}))$ or . . . (higher order teXs.)

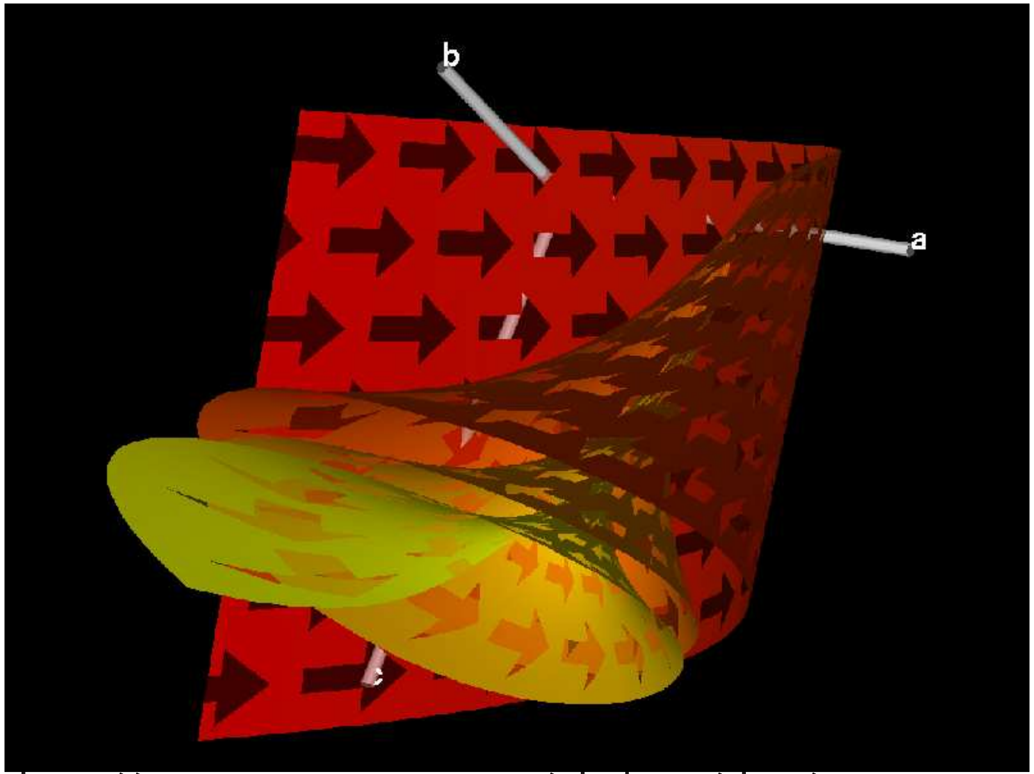


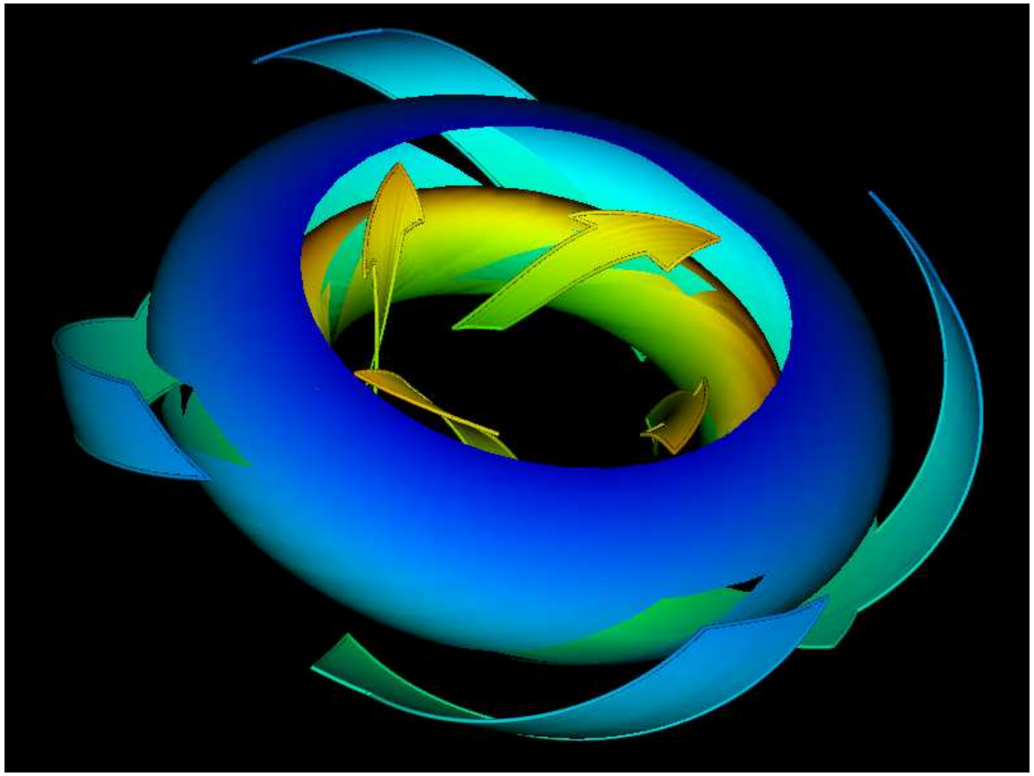
[Zöckler et al. '96]



[image courtesy by NASA, '98]

Stream arrows – enhancing stream surfaces





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Arrows indicate **flow direction** and **flow magnitude** (local properties), **flow convergence/divergence**, etc.

Approaching a dynamical system - some math

- **Information on variables, parameters –** values just positive?! special relations?

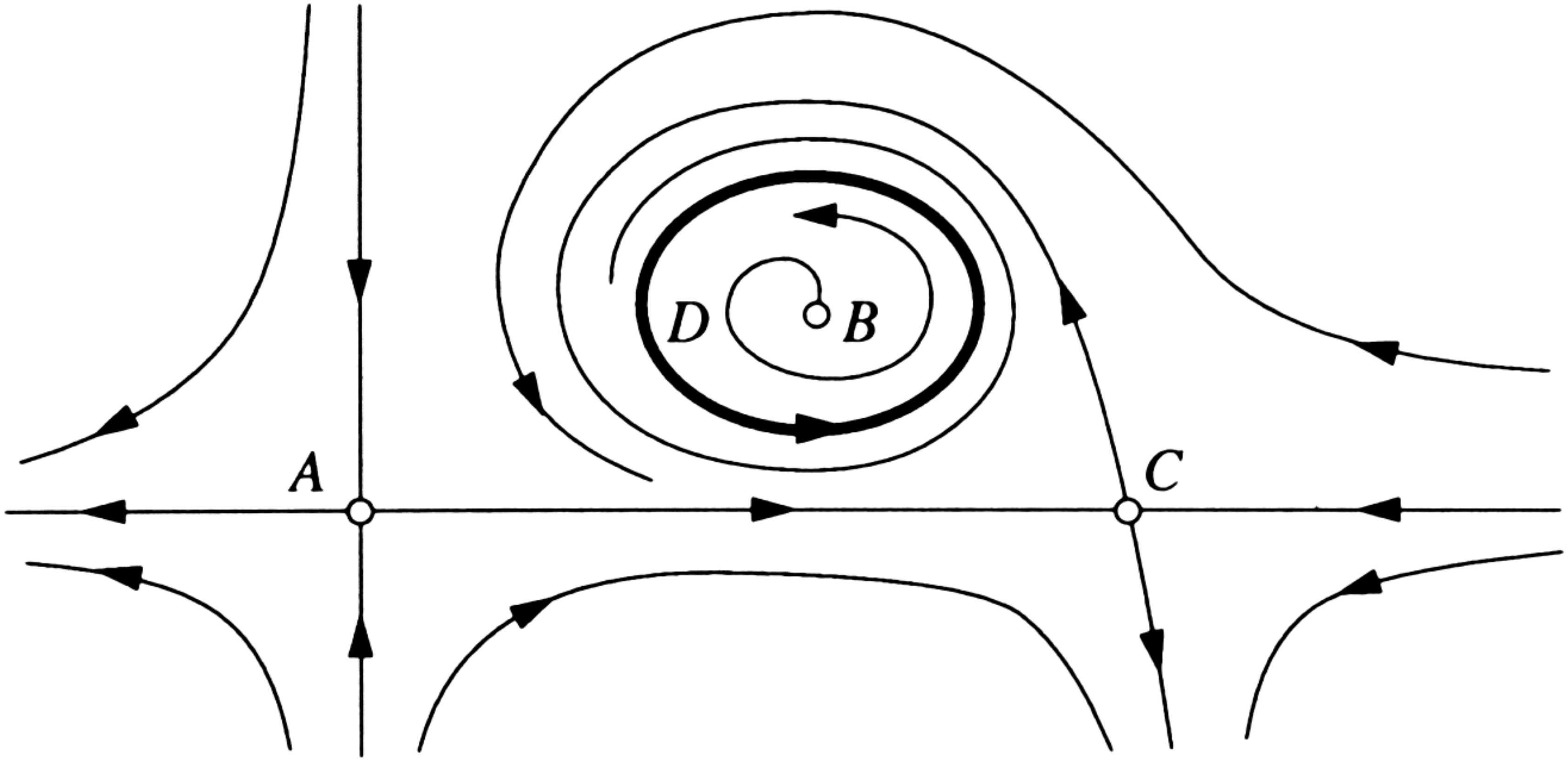
- **Structures of prime interest: equilibria –**
  (1) investigating $\mathbf{f_p}(\mathbf{c}_i) = 0$ (no motion) yields **locations** of equilibria $\mathbf{c}_i$,
  (2) investigating Jacobian matrix $\nabla\mathbf{f_p}|_{\mathbf{c}_i}$ yields **first-order properties** of equilibria

- **Characteristic structures –**
  (1) loci: equilibria $\mathbf{c}_i$, cycles $\mathcal{C}_i(t)$ ($\mathcal{C}_i(t+T) = \mathcal{C}_i(t)$), tori, chaotic attractors, etc.
  (2) first-order properties: attractors, repellors, saddles

- **Further interesting structures –**
  eigen-manifolds, basins of attraction, separatrices, aso.

Result: a **topological skeleton** (plus attributes) of the dynamical system

Topological skeleton – example [Strogatz '94]



Depicted: three equilibria, one cycle, and flow on characteristic structures

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Lotka & Volterra (extended) – equilibria

Searching the so-called **iso-clines** (prey now follows logistic growth) . . .

$$\begin{aligned} \dot{x} &= r \cdot x \cdot (1 - \tfrac{x}{k}) - p \cdot x \cdot y = 0 \quad \ldots \quad \text{“no change in amount of prey”} \\ \dot{y} &= e \cdot p \cdot x \cdot y - m \cdot y = 0 \quad \ldots \quad \text{“no change in number of predators”} \end{aligned}$$

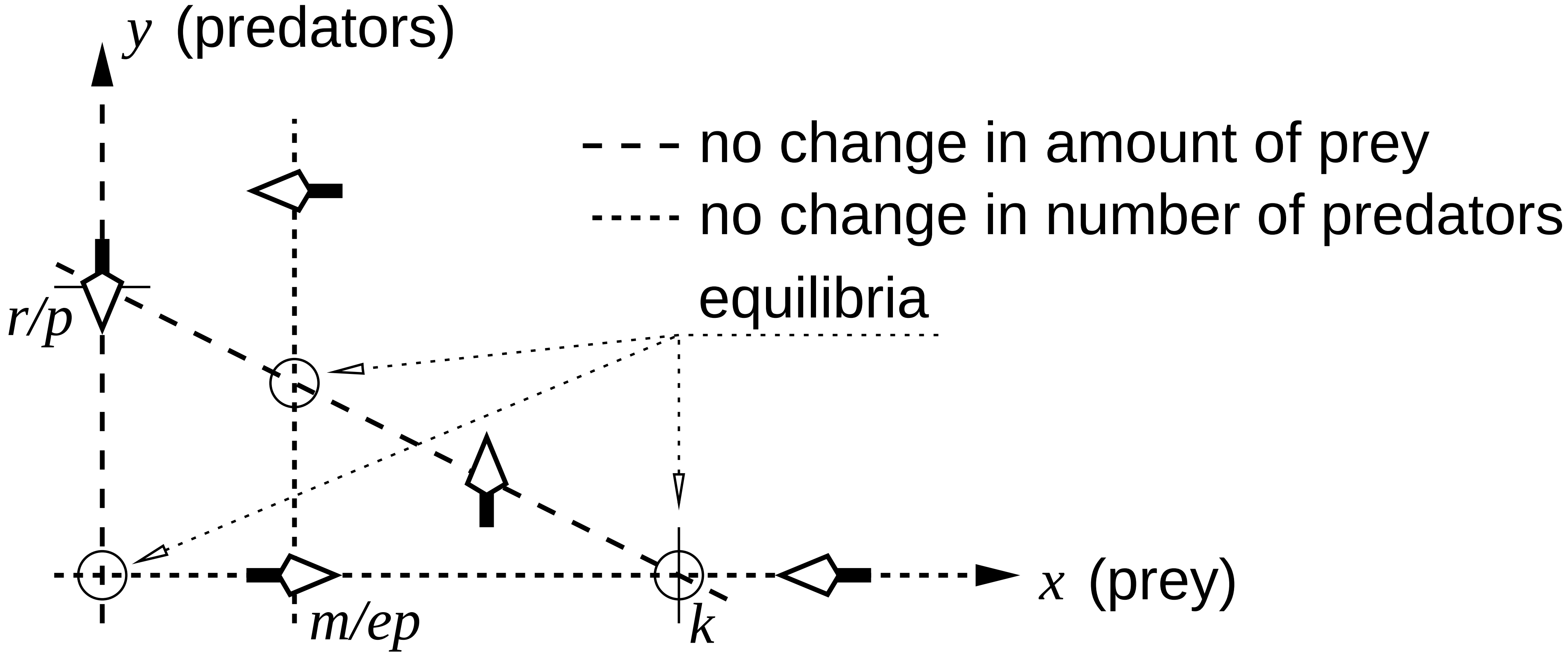
(1) $\dot{x} = 0 \;\rightarrow\; r \cdot x \cdot (1 - \frac{x}{k}) = p \cdot x \cdot y \;\rightarrow\; x = 0$ or $y = \frac{r}{p} \cdot (1 - \frac{x}{k})$

(2) $\dot{y} = 0 \;\rightarrow\; e \cdot p \cdot x \cdot y = m \cdot y \;\rightarrow\; y = 0$ or $x = \frac{m}{e \cdot p}$

Three equilibria: $\mathbf{c}_0 = (\,0 \;\; 0\,)^\top \quad \mathbf{c}_1 = (\,k \;\; 0\,)^\top \quad \mathbf{c}_2 = \left(\,\frac{m}{e \cdot p} \;\; \frac{r}{p} \cdot (1 - \frac{m}{e \cdot p}/k)\,\right)^\top$

Dynamics at $\mathbf{c}_0$: no prey, no predators $\rightarrow$ nothing to evolve . . .
Dynamics at $\mathbf{c}_1$: amount of prey stable, no predators $\rightarrow$ prey not disturbed (hunted) . . .
Dynamics at $\mathbf{c}_2$: equilibrium state $\rightarrow$ reproduction of prey equals loss (hunted),
mortality of predators equals reproduction

Lotka-Volterra 2 – iso-clines, equilibria



Depicted: flow at iso-clines and domain boundary, equilibria

Note: relation $m/ep \overset{?}{\lesseqgtr} k$ becomes important – one equilibrium depends on $m/ep < k$!

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Local properties of Lotka-Volterra 2 equilibria

Jacobian matrix at $\mathbf{c}_0$: $\nabla\mathbf{f}|_{\mathbf{c}_0} = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}$ $\Rightarrow$ $\mathbf{c}_0$ is a saddle!

$\rightarrow$ states $\mathbf{x}$ depart from $\mathbf{c}_0$ along the x-axis $(\lambda_1 = r > 0)$

$\rightarrow$ states $\mathbf{x}$ approach $\mathbf{c}_0$ along the y-axis $(\lambda_2 = -m < 0)$

$\mathbf{J}$ at $\mathbf{c}_1$: $\nabla\mathbf{f}|_{\mathbf{c}_1} = \begin{pmatrix} -r & -pk \\ 0 & epk-m \end{pmatrix}$ $\Rightarrow$ $\mathbf{c}_1$ is an attractor or saddle!

$\rightarrow$ states $\mathbf{x}$ approach $\mathbf{c}_1$ along the x-axis, $(\lambda_1 = -r < 0)$

$\rightarrow$ states $\mathbf{x}$ approach $\mathbf{c}_1$ (depart from $\mathbf{c}_1$) if $\frac{m}{ep} > k$ $(\frac{m}{ep} < k)$ $(\lambda_2 = epk-m \overset{?}{\lessgtr} 0)$

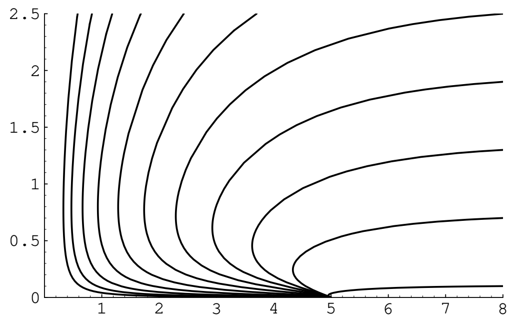
$\nabla\mathbf{f}|_{\mathbf{c}_2} = \begin{pmatrix} -r/k \cdot m/ep & -p \cdot m/ep \\ er/k \cdot (k-m/ep) & 0 \end{pmatrix}$ $\Rightarrow$ $\mathbf{c}_2$ is a focal node!

$\rightarrow$ states $\mathbf{x}$ approach $\mathbf{c}_2$ while rotating around the focal node $(Re(\lambda_{1,2}) < 0)$

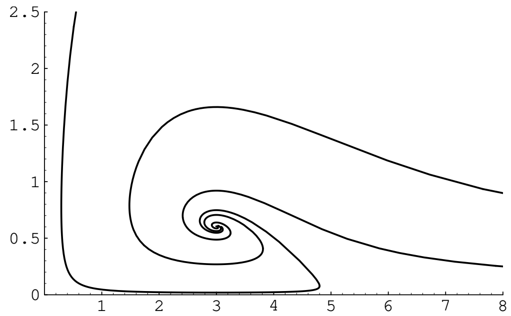
Lotka-Volterra 2 – finishing up math

Depending on relation $m/ep \overset{?}{\lessgtr} k$ we see two scenarios of evolution:

Predators do not survive at all ($m/ep > k$) –
Due to an insufficient reproduction efficiency, the only stable equilibrium is c_1, i.e., prey lives at its carrying capacity k, no predators alive ($y = 0$).



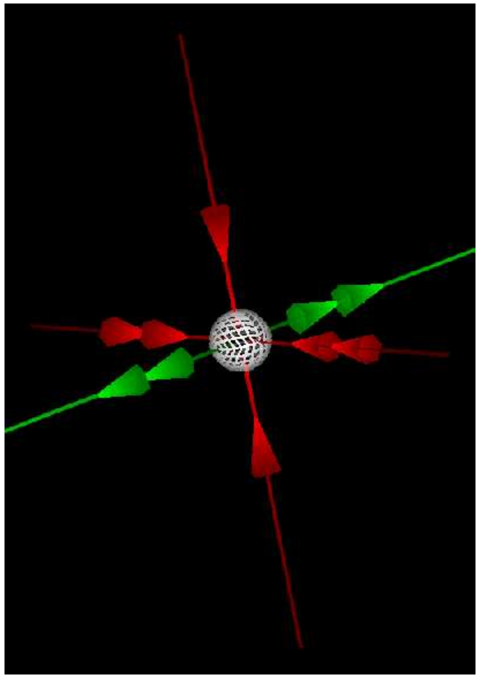
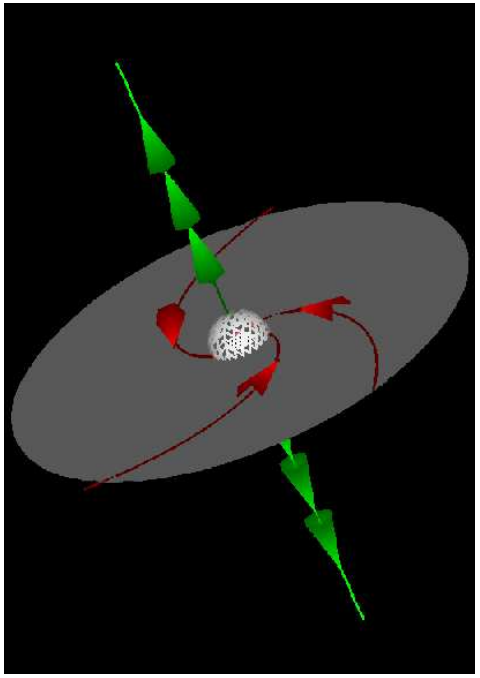
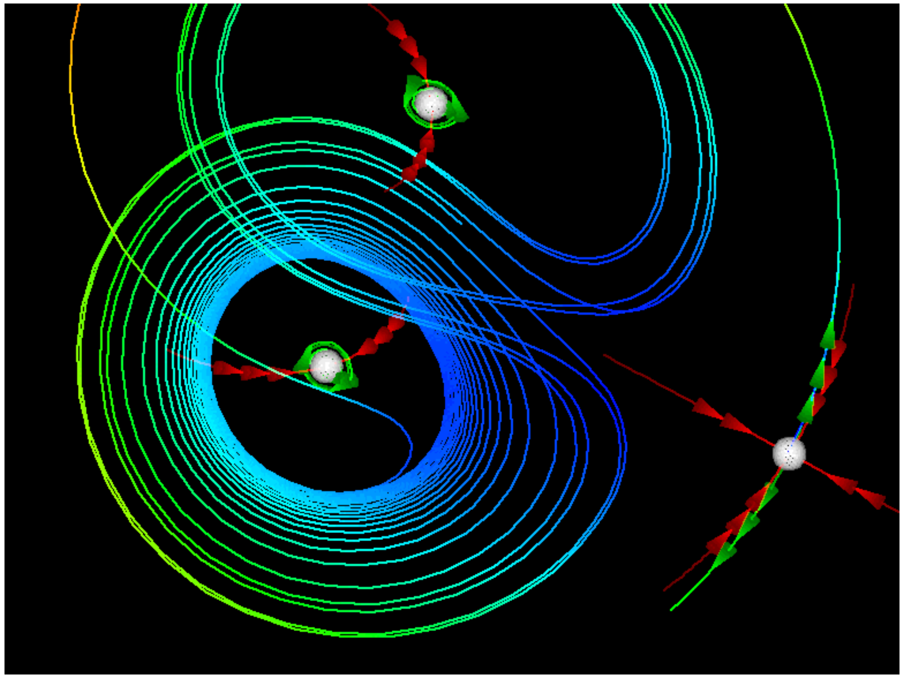
Stable coexistence predators / prey ($m/ep < k$) –
An oscillating amount of prey as well as an oscillating population of predators (phase-shifted) converges towards a stable state of coexistence ($x < k$, $y = \frac{r}{p}(1 - x/k)$).



In between: a so-called **bifurcation** occurs.

... following: visualization again!

Equilibria and skeleton visualization

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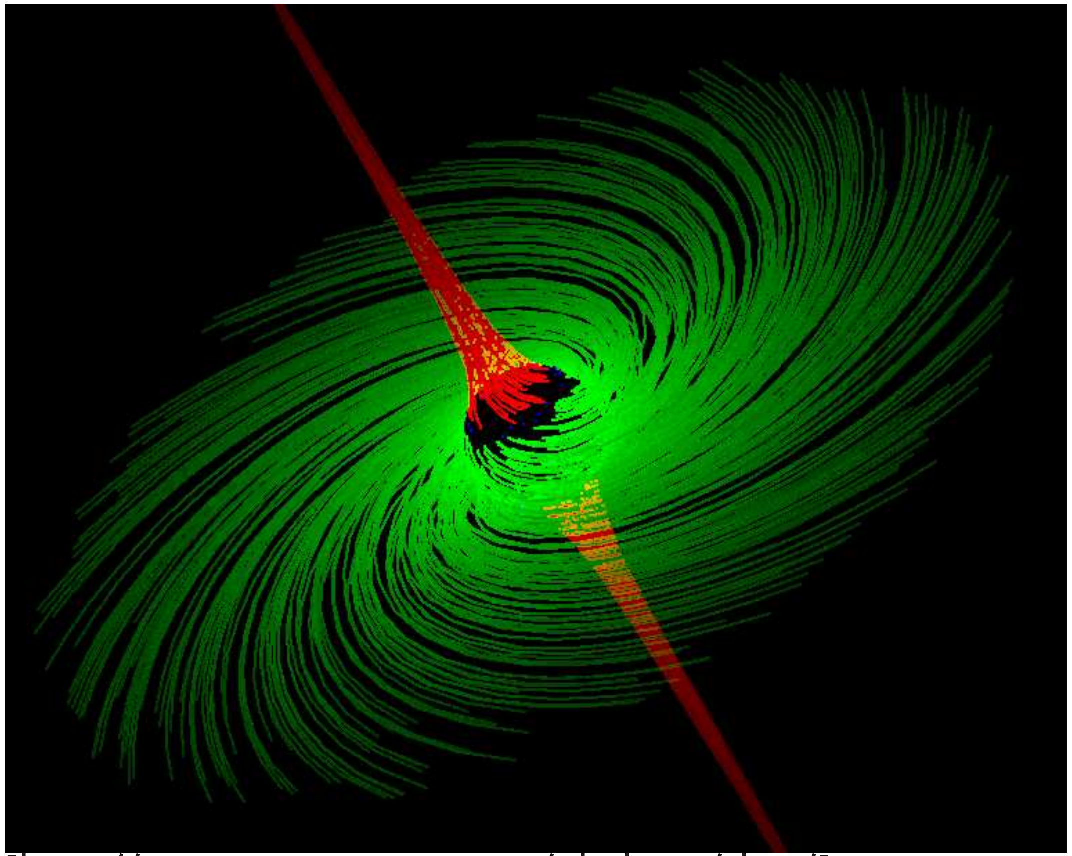
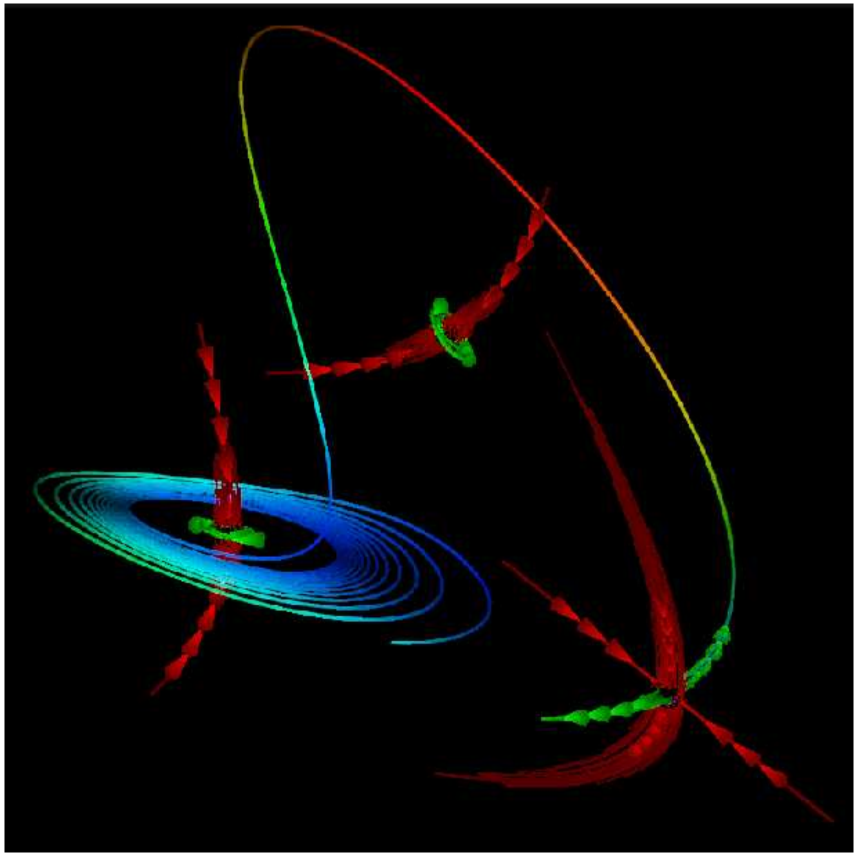
Depicted: two saddle nodes and equilibria of Lorenz system

Enhanced skeleton visualization





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Streamlets near saddle and equilibria of Lorenz system (enhanced by streamlets)

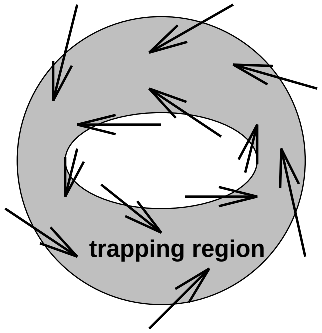
Dynamical system math, next level

- **Cycles –** usually difficult to indentify/locate! Just a few notes here . . .

  One important step is to know about non-existing cycles, for example:
- **Gradient systems (can be written as $\mathrm{f}(\mathrm{x}) = \nabla v(\mathrm{x})$) –** no cycles possible!

  Proofing the existence of a cycle, for example:

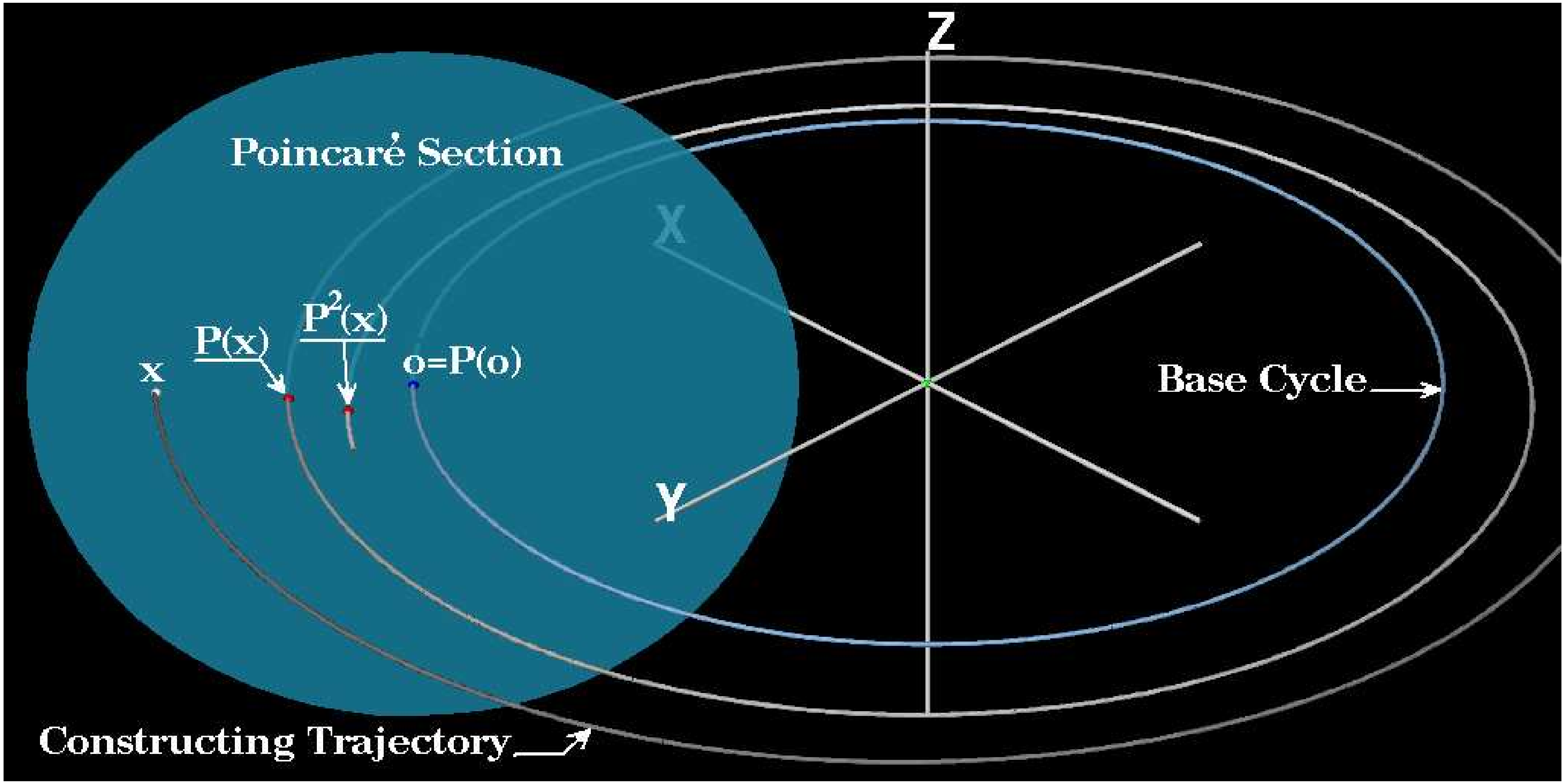
  Poincaré-Bendixson theorem: use so-called **trapping region** (sub-set of phase space, incl. infinitely long trajectories, no equilibria) $\rightarrow$ cycle must exist inside!



- **Tori, strange attractors –** even more difficult

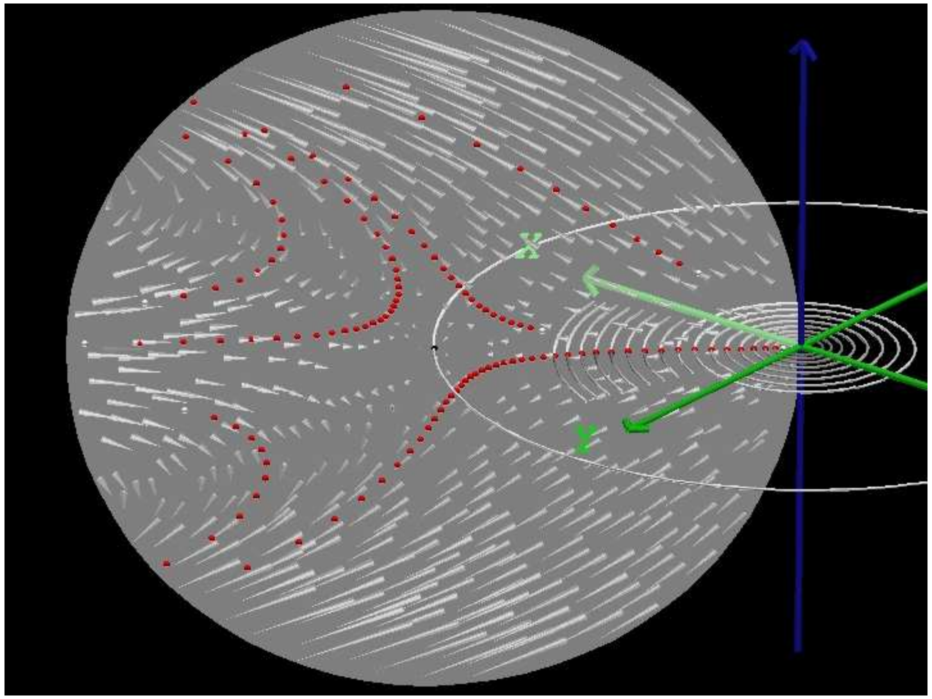
$\Rightarrow$ **numerical methods** are often facilitated

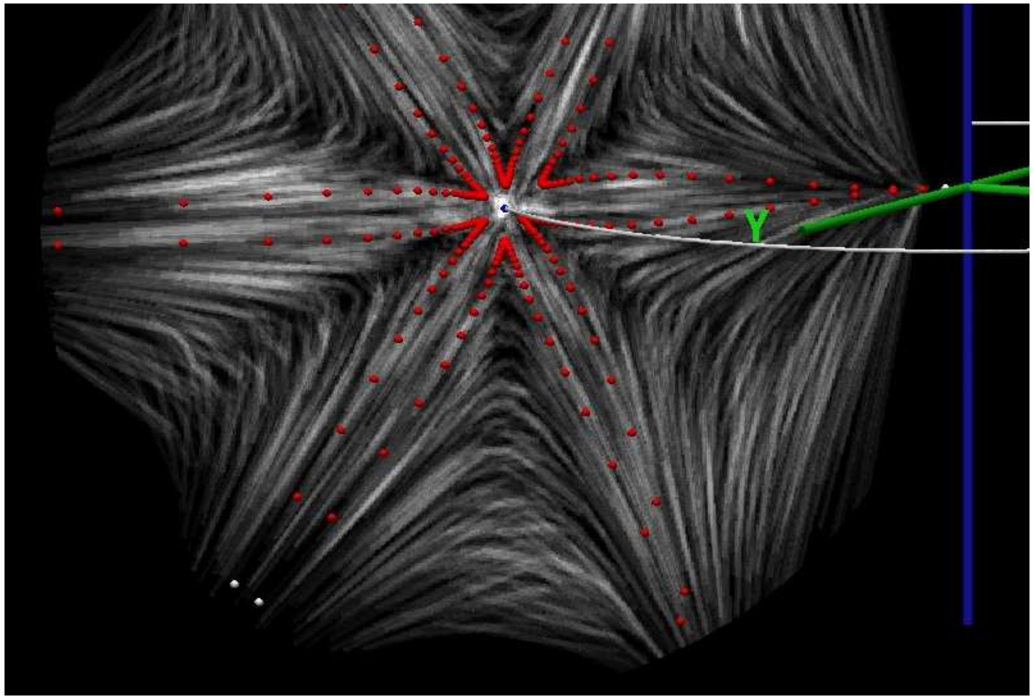
Periodic systems and Poincaré sections



Poincaré section $\rightarrow$ relation $\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x})$ (continuous) $\leftrightarrow$ $\mathrm{p}_{n+1} = \mathrm{p}(\mathrm{p}_n)$ (discrete)

Poincaré maps and visualization 1/2





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Depicted: Poincaré map visualization plus 3D cues, spot noise for Poincaré map

Poincaré maps and visualization 2/2

Poincaré map vis. . . .

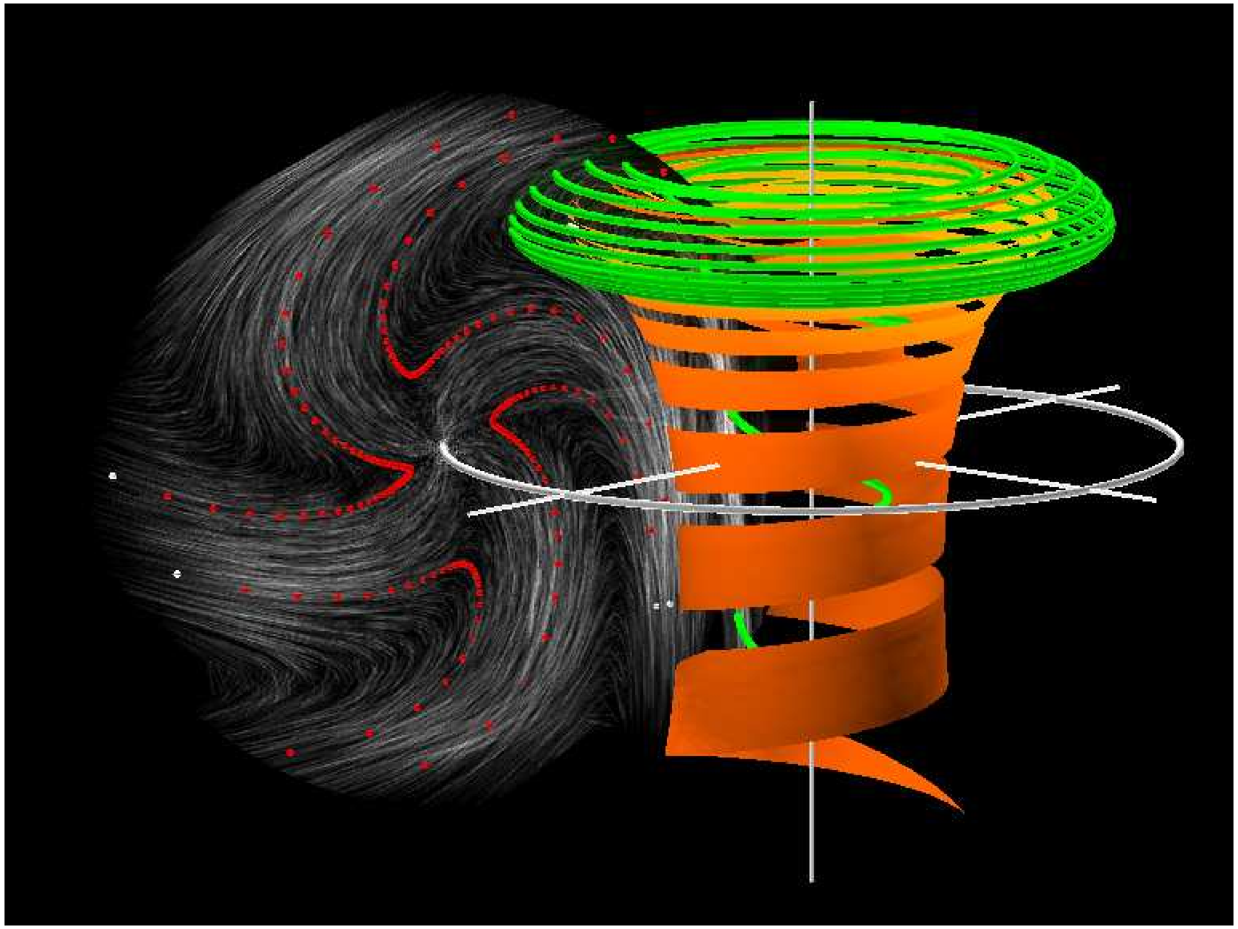
. . . and (!) . . .

. . . **direct visualization**

Spot noise
orbit vis.

$+$

stream lines
stream surface



Conclusions, acknowledgements, URL

- **Math –** very important to understand dynamical systems $\rightarrow$ **analysis**!
- **Stream lines, stream surfaces –** very intuitive $\rightarrow$ **exploration**!
- **Enhanced skeleton visualization –** dense and compact representation, incl.:
  - **topological skeleton** (or sub-set of) and . . .
  - its **geometric structure** (numerically approximated), plus
  - **intuitive flow visualization cues** for deeper insights

Thanx 2 Helmut Doleisch, Thomas Kučera, Lukas Mroz,
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and: **Thank you!**

http://www.cg.tuwien.ac.at/~helwig/diss/