Classifying the Visualization of Analytically Specified Dynamical Systems

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Abstract. In this paper we suggest a classification of visualization techniques for analytically specified dynamical systems into four different approaches. We distinguish between *local properties*, the *topology of behavior*, *global properties*, and *classes of dynamical systems* with respect to various topics of visualization. By presenting advanced visualization techniques that we applied during three recent projects, we discuss their embedding within the classification scheme. The dynamical systems visualized are the "Dynastic Cycle", which is a model for rise and fall of dynasties in ancient China, the "Wonderland" model, that simulates the interactions of population growth, economic activities, and environmental pollution, and a model for mixed-mode oscillations, which occurs in chemistry.

Key words:visualization, dynamical systems

1. Introduction

Scientists in areas like mathematics, physics, bio-chemistry, economics, medical research, sociology, and many others, often investigate dynamical systems as models of natural phenomena. Any set of variables, whose configuration describes the state x of a model at a specific point in time, is called a *dynamical system*, if its evolution in time is determined by the current state of the system and by a time-dependent input-function u. See Figure 1 for an illustration of this principle. One main characteristic of a dynamical system is whether it is *continuous* or *discrete*. The rules determining the evolution of a continuous dynamical system (often called *flow*) are given as a set of differential equations ($\dot{x} = f(x)$), whereas the rules for a discrete dynamical system (also called *map*) are difference equations ($x_{n+1} = f(x_n)$) [6].

Another important characteristic of a dynamical system is whether f is time-dependent or not (*unsteady* vs. *steady* models). For analysis issues the classification into *linear* and *non-linear* dynamical systems is also very important. Complex behavior is typical for non-linear systems, but linear systems are nevertheless very important. Most of the techniques for the analysis of non-linear systems are based on linearization and subsequent investigation of the linearized models.

The dimension of a dynamical system is given by the minimum number of variables, which describe its state entirely. By associating each state variable with a coordinate axis the *phase space* is constructed. Each point in phase space represents a state of the system at a specific point in time. A *trajectory*, i.e. a curve in phase space, describes the evolution of an initial state in a continuous dynamical system. For discrete models such an evolution is given by an *orbit*, i.e. a set of points in phase space.

The investigation and exploration of mathematical models can be supported by visualization providing helpful insights to the dynamics of dynamical systems. Various techniques have been developed in this field in the past years [1][2][8][10][11]. In section 2 we suggest a classification for the visualization of dynamical systems into four different approaches. In section 3 we present a set of advanced visualization techniques that were applied to three particular dynamical systems during recent projects. Section 4 gives a short summary.

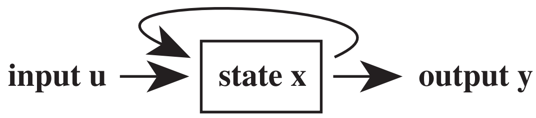

Fig. 1. principal structure of a dynamical system

2. Four Approaches to Visualizing Dynamical Systems

Visualization has become an important field of research in the past decade. Today a number of separate fields can be distinguished, e.g., volume visualization, flow visualization, and many others. One of these fields the visualization of dynamical systems subsumes a lot of techniques depending on the topic of interest, e.g., local properties or long-term behavior. In the following sections four classes of techniques in this field of visualization are presented.

As one approach (section 2.1) visualization is intended to represent *local properties* at some point (state) of the system. Glyphs are a prominent example for visualization techniques of this kind [1][6][7][12]. Another approach (section 2.2) is to represent the *topology of behavior* of a dynamical system. The visualization of characteristic subsets, e.g., attractors, helps to understand the global behavior induced by the corresponding system [3]. Other techniques try to directly visualize the behavior of the model in a global fashion. Arrow plots, streamlines, streaklines, and many other techniques belong to this approach (section 2.3) [4][8][9]. The last approach (section 2.4) presented is the visual representation of an entire *class of dynamical systems*. Bifurcation diagrams are probably the most notable technique in this subfield of visualization [3].

2.1. Local Properties

Often scientists want to analyze their dynamical system at some specific point in phase space. They are interested in local aspects as, for example, velocity, acceleration, and divergence. A vortex or a point near an obstacle may be such a point of special interest. Thus visualization techniques were developed that help to investigate these local flow properties.

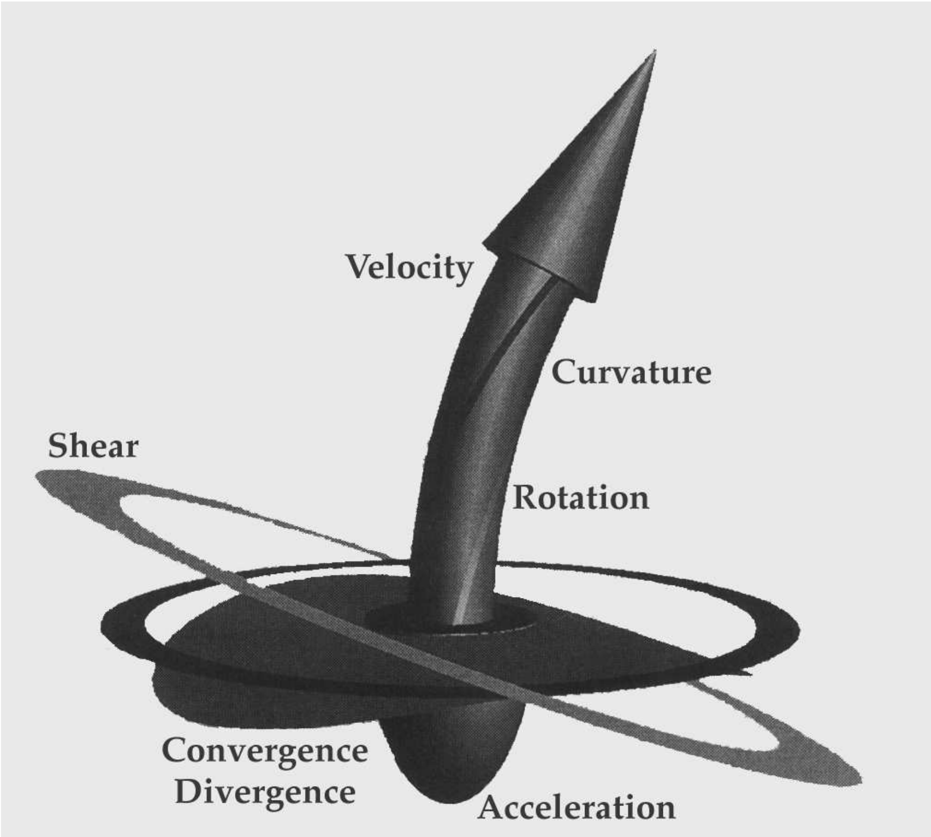

Fig. 2. de Leeuw, van Wijk [7]: a glyph visualizing local properties

De Leeuw and van Wijk use a *glyph* to visualize local flow behavior [7]. They decompose the velocity gradient tensor into components parallel and perpendicular to the flow. This is done after the system is transformed it into a local coordinate system. Components parallel to the flow are acceleration, shear, and curvature. Torsion and convergence/divergence are the components perpendicular to the flow. All these local attributes—parallel as well as perpendicular to the flow—are represented by separate geometric elements that are combined in the visual representation.

The glyph mainly consists of a 3D arrow, which represents the velocity, its direction, the curvature and the rotation of the flow. Additional elements are added to visualize the shear component, acceleration and the convergence or divergence in the flow. See Figure 2 for a sample glyph of this kind. In the implementation of de Leeuw and van Wijk the glyph can be located interactively at interesting points in the flow or positioned automatically, for example, at consecutive points on a streamline.

Streamlines are a commonly used method for visualizing constant vector fields or snapshots of a time-dependent flow [8]. Schröder, Volpe, and Lorenson propose to use an n-sided polygon positioned perpendicular to the flow at specific points of a streamline to visualize local attributes [1]. With this *streampolygon* a visualization of local deformations due to a rotation around the flow vector and both normal and strain shear is possible.

Extending the principle of streamlines and streamribbons, the streampolygon can be swept along the streamline (forming a *streamtube*), while its shape is transformed according to the local flow characteristics. Due to its two-dimensional cross-section more local information can be visualized using such a streamtube.

Local stability characteristics are often an especially important information when investigating a dynamical system. Stability depends on whether states near a point of interest converge or diverge during evolution of the vector field. Several techniques to visualize these local characteristics in a dynamical system have been developed at our department [12]. We use, for example, spheres and cylinders along a trajectory of a dynamical system to represent points of convergence and/or divergence. As another method we use a technique similar to the streampolygon to visualize the evolution of trajectories close to a specific trajectory.

Mathematicians often use Poincaré sections to investigate chaotic and/or strange attractors of dynamical systems [6]. Poincaré sections reduce the dimension of a dynamical system and allow to analyze the nature of the attractor more easily. These techniques are well suited for visualizing dynamical systems [10].

2.2. The Topology of Behavior

As already mentioned in section 1 the behavior of a dynamical system can be characterized by the interpretation of the topological structure of special subsets. Generally such characteristic subsets are representatives out of a well-defined set of possible geometrical structures, e.g., points, curves, surfaces, and others. The model's behavior near such elements is used to further subdivide this set of structures. Attracting, repelling, and saddle entities can be distinguished. Additionally other subsets connecting attractors, repellors, and saddles, belong to this class of characteristic topological elements, e.g., separatrices, saddle-connections, homo-clinic orbits, and others [14].

Unfortunately, most of these special subsets are not very easy to identify automatically and computational approaches are rare in this field. This could have been a reason

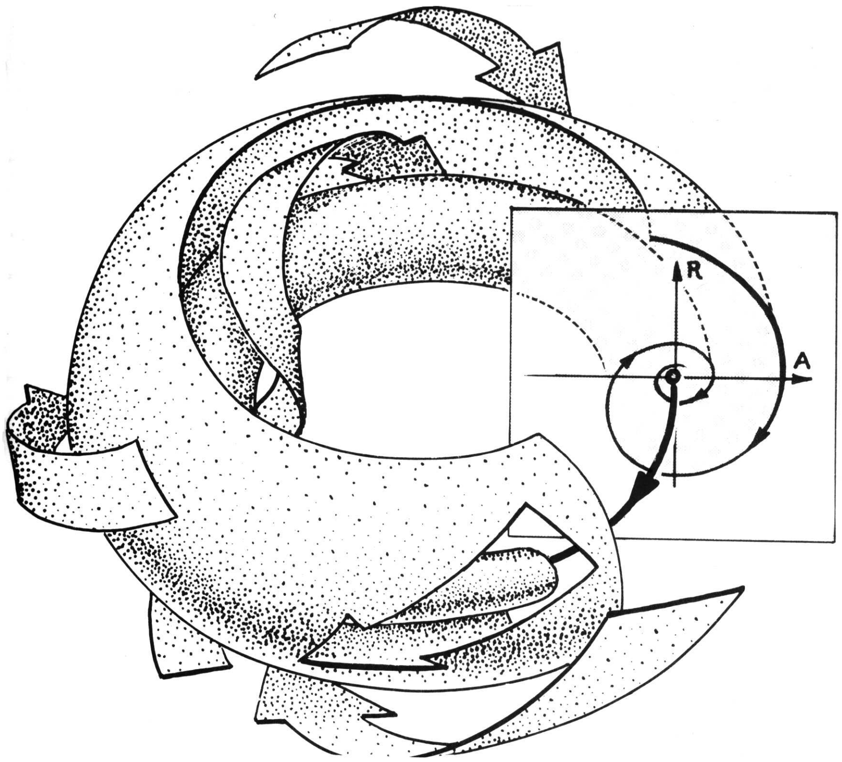

Fig. 3. Abraham, Shaw [3]: visualizing the topology of behavior

why Abraham, a mathematician, and Shaw, an artist, worked together and came up with an unconventional book "Dynamics – The Geometry of Behavior" [3]. Hand-drawn illustrations give a fascinating introduction to topological aspects of dynamical systems. The mathematician's knowledge of the above mentioned special subsets and the ability of an artist to visualize only important aspects give an optimal combination to simplify the understanding of the behavior of dynamical systems. Refer to Figure 3 for a typical image out of their book. Some of Shaw's ideas have been integrated into a general purpose visualization system at our department [16]. See section 3.3 for details.

2.3. Global Properties

Most approaches for visualizing the global behavior of a dynamical system (e.g., streamlines, pathlines, streamsurfaces, etc.) rely on standard methods from the field of experimental fluid flow research (e.g., injecting some drops of dye into the flow). There are several implementations of such techniques given in literature [10].

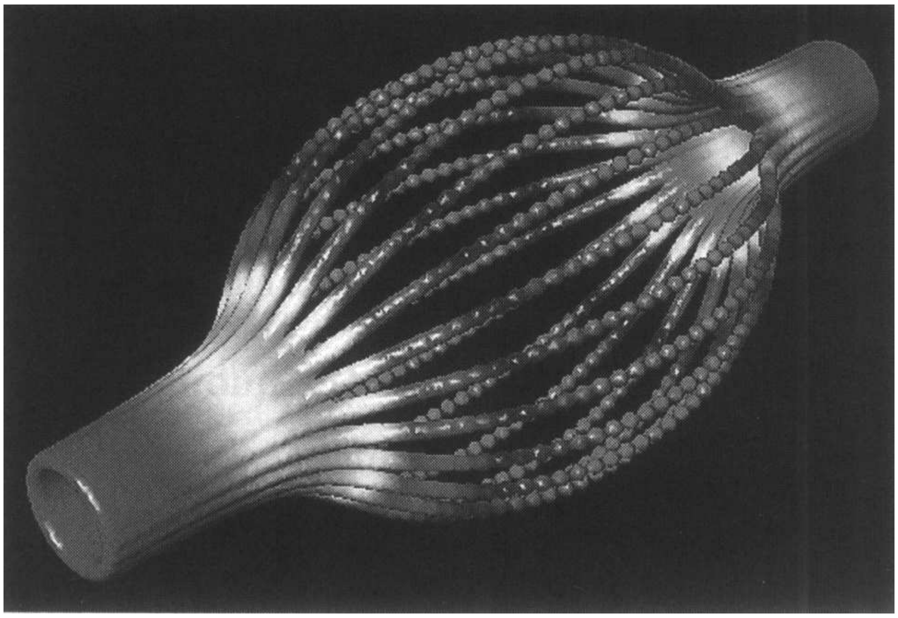

Fig. 4. Brill et al. [9]: streamballs flowing around an obstacle

As another notable technique, Brill et al. introduce the concept of streamballs [9], which is based upon implicit surface generation techniques, and which extend the methods of streamlines and streamsurfaces (see Figure 4). Located along such a streamline, or spread over a streamsurface, near streamballs blend smoothly to form one implicitly defined surface.

The streamball approach leads to good results even if the underlying dynamics are complex. Because of the blending feature of streamballs, detecting the regions of divergence or convergence using this method is easier than with standard visualization methods (e.g., timelines). When a continuous skeleton is used instead of a discrete set of centerpoints, the generated convolution surface can be used to visualize tensor fields.

2.4. Classes of Dynamical Systems

The most commonly used algorithm for visualizing an entire class of dynamical systems is plotting its bifurcation diagram. Such a diagram is constructed by extending the plot of the dynamical system's long-term behavior by additional coordinate axes corresponding to various system parameters. Variations along these axes represent modifications to the model and thus an entire set of dynamical systems can be represented by such a bifurcation diagram. A software system for the visualization of bifurcation diagrams was developed at our department [10].

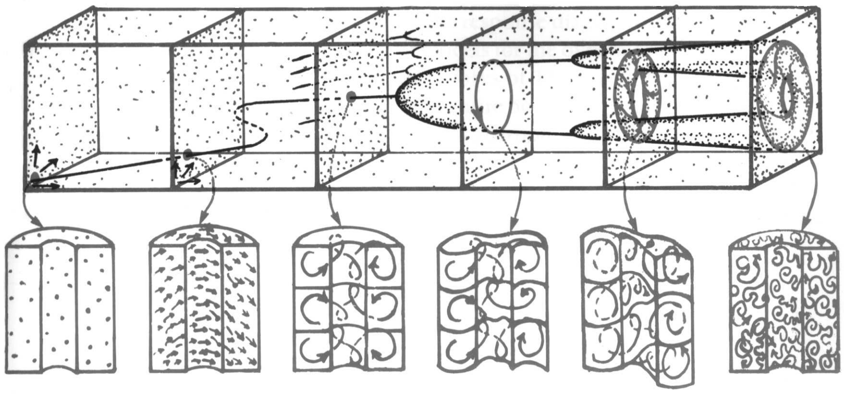

Fig. 5. Abraham, Shaw [3]: hand-drawn bifurcation diagram

Typically such a bifurcation diagram is two-dimensional, which means, that one axis is used for representing the system's long-term behavior and the other axis is used for variations of a system parameter. Using two dimensions for the representation of the dynamical system's behavior and one for the change of a parameter gives a 3D diagram, which seems to be helpful for analyzing complex systems. Good examples are given again by Abraham and Shaw (see Figure 5) [3].

3. Recent Visualization Projects

Several visualization projects have been carried out at our department which took into considerations the requirements of various scientists investigating dynamical systems. The visualization of the "Dynastic Cycle" is presented in section 3.1 [15], the "Wonderland" model and a set of advanced visualization techniques, that were used with this dynamical system, are discussed in section 3.2 [13][17], and finally, in section 3.3, the visualization of a model of mixed-mode oscillations using a technique called streamarrows is discussed [16]. The projects are presented by discussing their embedding within the classification of section 2 and by summarizing the results of these projects.

3.1. The Dynastic Cycle

The first project presented in this section deals with the visualization of the "Dynastic Cycle"—a three-dimensional dynamical system, that was modeled to give an explanation for the rise and fall of dynasties in ancient China, given as alternating periods of anarchy and despotism. The three system variables X, Y, and Z express the amount of farmers, bandits, and soldiers, respectively. The model defines their interactions similarly to well-known food-chains (prey, predator, and super-predator). The evolution induced by the Dynastic Cycle is governed by so-called slow-fast dynamics. Two of the system variables (X, Y) are *fast* variables that change rapidly in comparison to the *slow* one (Z). The knowledge about this slow-fast characteristics simplifies the analysis and must be considered during visualization.

The goal of the project [15] was to visualize the long-term behavior exhibited by this dynamical system in phase space. The techniques we chose for this purpose were therefore taken from the class "global properties" (see Figure 6). Furthermore the scientists we cooperated with were also very interested in an entire class of Dynastic Cycle models, that is given by varying key parameters of this dynamical system. Even chaotic behavior can be found in this class of models (see Figure 7). Thus the distinction between different types of qualitative behavior was another important aspect. Due to these requirements we used the combination of the following techniques for visualizing this dynamical system:

- A streamline that starts at some arbitrary but representative state in phase space was assumed to be a suitable representation of the model's long-term behavior. It was visualized by sweeping a polygonal cross-section along the streamline forming a tube which follows the evolution of the initial state. This approach is similar to the streampolygon/streamtube by Schröder, Volpe, and Lorenson [1], but no local deformation of the cross-section was performed. A sufficiently big number of vertices was chosen for the polygonal cross-section to generate an impression of a cylindrical 3D-object representing the streamline.
- Due to the slow-fast nature of the dynamics of the Dynastic Cycle it was crucial to represent the velocity at every point of the streamline appropriately. Color coding was used for this purpose such that depending on the velocity the color of the streamline representation was chosen according to a user defined color scale. In Figures 6 and 7 color blue indicates slow motion, whereas yellow or red colors represent fast transitions.
- Additionally three-dimensional arrow-glyphs were positioned at certain points of the streamline to visualize the direction of the flow. Each arrow object is composed of parts of a cylinder and a cone. It is bent according to the shape of the streamline such that the transformed axes of cylinder and cone coincide with the trajectory.

Figure 6 shows a typical version of the Dynastic Cycle model. Particular states of special interest were labeled to support the interpretation of the model. A small number of soldiers indicates anarchy (states 2 and 3), whereas a lot of soldiers is representative for despotism (states 4 and 1). Labeled axes complete the representation of this dynamical system. Using color coding it is shown that, for example, during the transition from anarchy (state 3) to despotism (state 1) a number of fast prey-predator cycles are performed.

By varying system parameters the long-term behavior induced by the model changes. Figure 7 shows such an altered system, which exhibits chaotic behavior. Again color blue indicates regions of slow developments, whereas yellow and red are used in areas of rapid changes.

3.2. The Wonderland Model

"Wonderland" is the name of an econometric dynamical system that models the interactions between population growth, economic activities and environmental pollution [13][17]. It is a four-dimensional system with three slow system variables and one fast variable. Similarly to the Dynastic Cycle the slow-fast characteristics of the induced dynamics can be exploited to analyze the model. Mainly three scenarios, i.e. the "dream scenario" (clean environment forever), the "horror scenario" (total environmental crash), and the "escape scenario" (cycles of environmental degeneration and regeneration), can be distinguished depending mainly on the values of two system parameters.

To understand which of the scenarios will occur, it is crucial to know the geometry of this model. In particular the knowledge about the location and the form of one specific manifold is the key to characterize the future behavior. Thus, adaptations and applications of visualization techniques in class "topology of behavior" were assumed suitable for representing this dynamical system. Additionally, techniques from the class "global properties" were used to improve the expressiveness of the results. Due to the slow-fast nature of the dynamics exhibited by the Wonderland model special visualization techniques were developed:

- Flow velocity is an essential parameter when examining the Wonderland model. A special texture is mapped onto the tube representation of a streamline. This allows to visualize flow velocity in single images as well as in animation sequences. The texture generates spiraling coils along the entire tube, where the distance between adjacent coils correspond to local flow velocity. A non-linear mapping function was used in the texture mapping step to take account of the vastly varying velocities in this model. Animating the texture rotates the coils around the tube, which gives a good impression of the velocity in all regions of the streamline.
- The most important part of this project was the visualization of the critical manifold, which is given as a curved surface in three space. Several visualization techniques were combined to facilitate understanding. Semi-transparency was used for representing

the surface, since the behavior in front as well as behind the manifold had to be visible. Texturing together with shading were applied to improve the perception of the shape and the location of the surface. Color coding was used to represent a local attribute, i.e. the divergence/convergence with respect to the fast variable of the system.

- A particle system and animation were used to demonstrate at some specific point in time the system behavior near the critical manifold. Particles are evenly distributed on the critical manifold. An animation sequence is produced by allowing the particles to follow the instantaneous flow of the system. This clearly illustrates the slow-fast dynamics of the Wonderland model.
  Animated tufts in combination with color coding were used to represent past and future development of states at the manifold, while the manifold itself moves through phase space.

Figures 8 and 9 contain a textured, shaded, and semi-transparent manifold, and a textured tube representing the escape scenario. Three system variables have been represented directly as coordinate axes, whereas the fourth variable of the system has been neglected. Repelling portions—with respect to the fast (vertical) variable—of the manifold have been colored red, whereas attracting parts are displayed in green. Critical curves (separating attracting and repelling parts) are modeled as golden tubes.

Figure 8 represents the development of the system until its sudden environmental crash. Shortly after the streamline passes the semi-transparent manifold near Z_1, the system collapses. Figure 9 shows the system evolution up to a later state – the falling trajectory has been caught by the backwards-moving manifold, such that the environment is able to regenerate again (escape scenario). For further results of this project refer to the references [17].

3.3. Mixed-Mode Oscillations

Mixed-mode oscillations are a phenomenon quite often encountered in chemical systems [5]. They owe their name to alternating large and small amplitude heights that are observed in time series of such a system. A quite simple set of kinetic equations that exhibits mixed-mode oscillations is the three-dimensional autocatalator. Because of its simplicity it is well suited for the investigation of the behavior of this kind of dynamical systems.

In collaboration with mathematicians, the autocatalator model has been visualized (see Figures 10 and 11). Since the characteristics of the long-term behavior induced by this model was the major point of interest, visualization techniques from the class "global properties" were chosen for representing the system. A typical streamsurface was computed. Evolving under the application of the dynamical system it forms a curled, shell-like structure. Almost totally independent from the viewing direction, the streamsurface occludes major parts of itself. To deal with this problem of occlusion the standard visualization technique of using streamsurfaces was extended by a set of special features [16]:

- The main extension to standard streamsurface techniques was the introduction of streamarrows. A streamsurface is segmented into a set of arrow-shaped objects and the remaining surface part. By assigning a certain level of semi-transparency to the streamarrows the viewer sees through the "holes" in the streamsurface and gets more information about the structure of the model. In addition to that, the use of streamarrows allows to visualize additional local information that is available on the streamsurface, i.e. direction of evolution, velocity, and divergence or convergence.
- As another augmentation of the visualization process we used anisotropic spot noise [2] as a texture to represent the evolution of states within the streamsurface. We used a hash-shaped spot to emphasize u- and v-lines of the generated texture. This results in a simultaneous representation of stream- and time-lines within the textured streamsurface.
- Using simple geometric objects such as planes to selectively cut away some parts of the system representation enables deeper insights into the "inner" regions of the model. All parts on one side of a properly positioned "cut plane" were assigned to be almost transparent allowing the viewer to see through and obtain a better image of the portions beyond the cut plane.
- The entire visualization was enriched by the application of animation techniques to several elements of the system representation. The construction of the streamsurfaces was animated simulating the behavior induced by the dynamical system. Streamarrows were animated to move within the streamsurface according the system behavior. In another animation sequence the cut plane was moved through the model providing an intuitive understanding of cutting away certain portions of the model.

Figure 10 shows a streamsurface that represents the system behavior in a particular region of phase space. The technique of streamarrows has been applied to allow viewers to imagine the shape and location of the streamsurface even in the inner portions of the shell-like structure. Labeled coordinate axes help to embed the representation within the semantical setting. Colors are used to distinguish between older and newer states of the streamsurface.

Refer to Figure 11 for an image that was taken from one of the produced animation sequences. It shows a visual representation of the autocatalator system that is segmented by the use of a cut plane into three portions. Entirely semi-transparent parts in the foreground (in front of the cut plane) are almost invisible to allow the viewer to see through. In the background (beyond the cut plane) streamarrows are shown on a streamsurface that has been textured using the anisotropic spot noise texture. In between those major parts, the cross-section of the cut plane and the model has been visualized by representing the resulting spatial curves as white tubes.

Note that a second streamsurface is included that represents the behavior of the *inverse* system, which is defined by the same equations as the original autocatalator, but with the assumption that time is running backwards. Both streamsurfaces intersect in a complicated manner. The application of advanced visualization techniques, however, easily illustrate the complex intersection patterns. For further results of this project refer to the references [16].

4. Summary

We presented a classification for visualization techniques in the field of analytically specified dynamical systems, which distinguishes between four different approaches, i.e. visualizing *local properties*, the *topology of behavior*, *global properties*, and entire *classes of dynamical systems*. For all these approaches some examples from literature were given and various projects, realized at our department were treated.

Three projects were discussed in detail, namely the visualization of the Dynastic Cycle, the Wonderland model, and a model of mixed-mode oscillations. Techniques that we applied to visualize these dynamical systems were discussed with respect to their embedding within the classification. Representative results were briefly discussed for all these projects.

Visualization offers valuable tools in analyzing complex dynamical systems. Depending on the specific requirements of a scientist suitable techniques can be selected from a variety of available visualization algorithms. Existing algorithms are adapted according to user specification, or new methods are developed. After various visualization projects the experience clearly indicates, that with visualization a faster and better understanding of the underlying dynamics of a dynamical system is gained.

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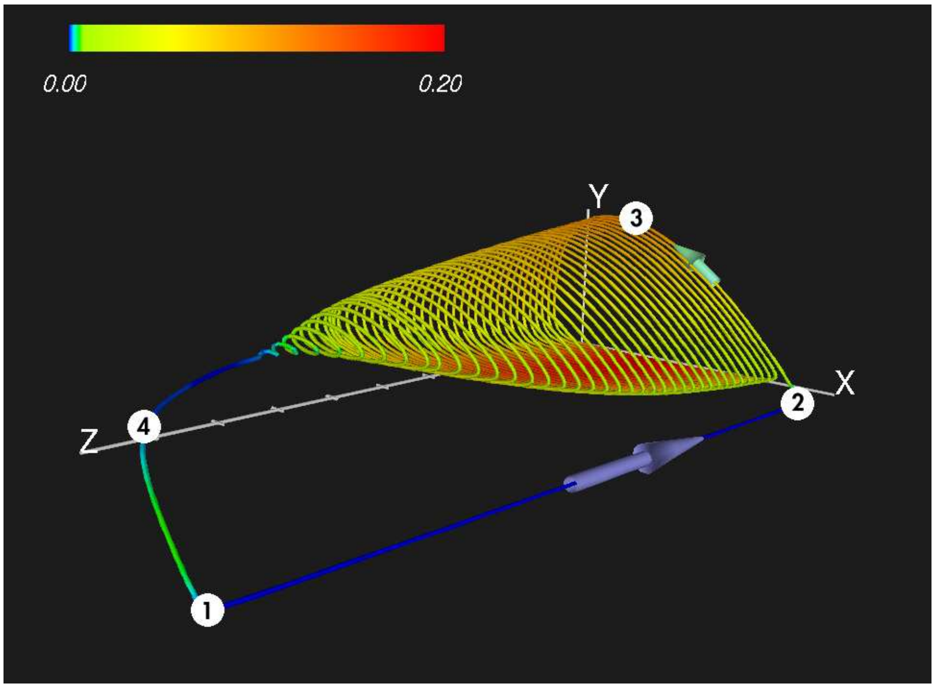

Fig. 6. A typical trajectory of the Dynastic Cycle

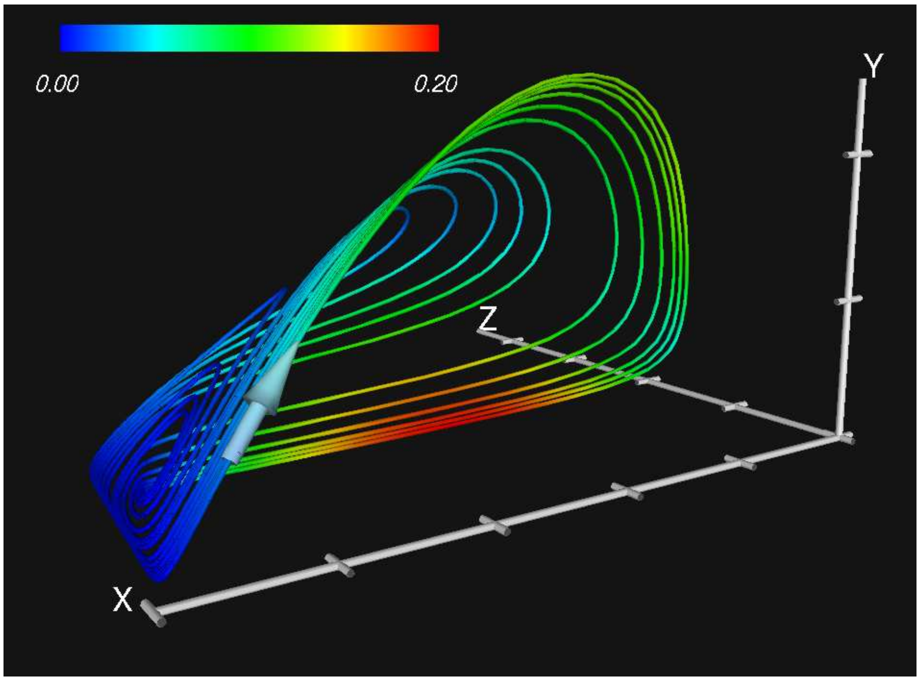

Fig. 7. A chaotic trajectory of the Dynastic Cycle

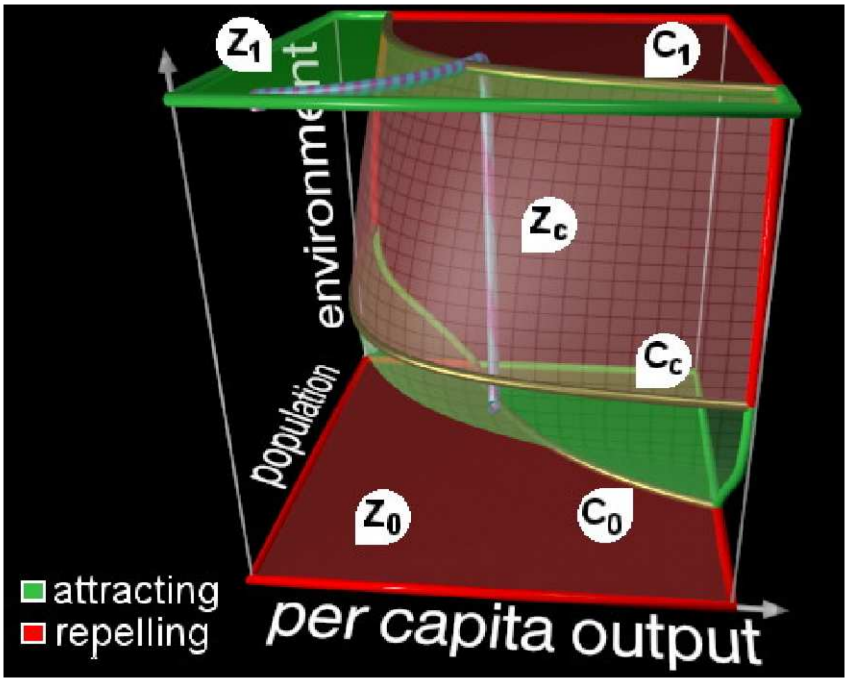

Fig. 8. Escape scenario, earlier stage (Wonderland model)

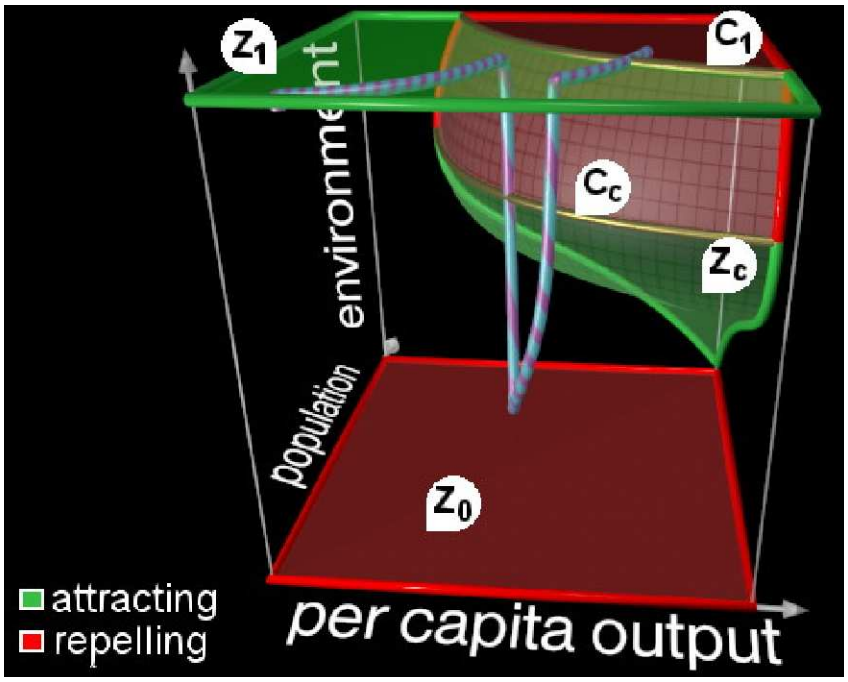

Fig. 9. Escape scenario, later stage (Wonderland model)

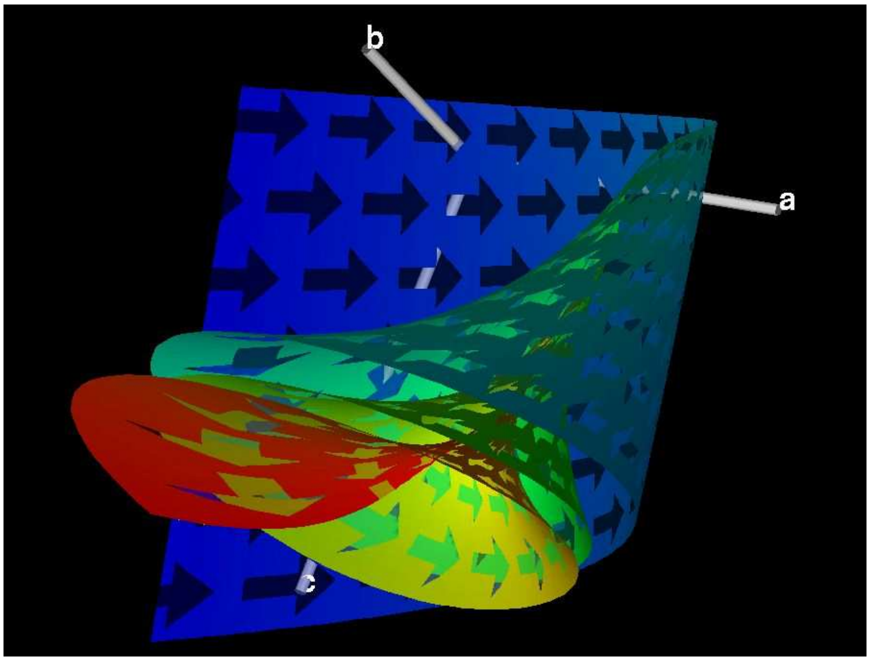

Fig. 10. Streamsurface with streamarrows

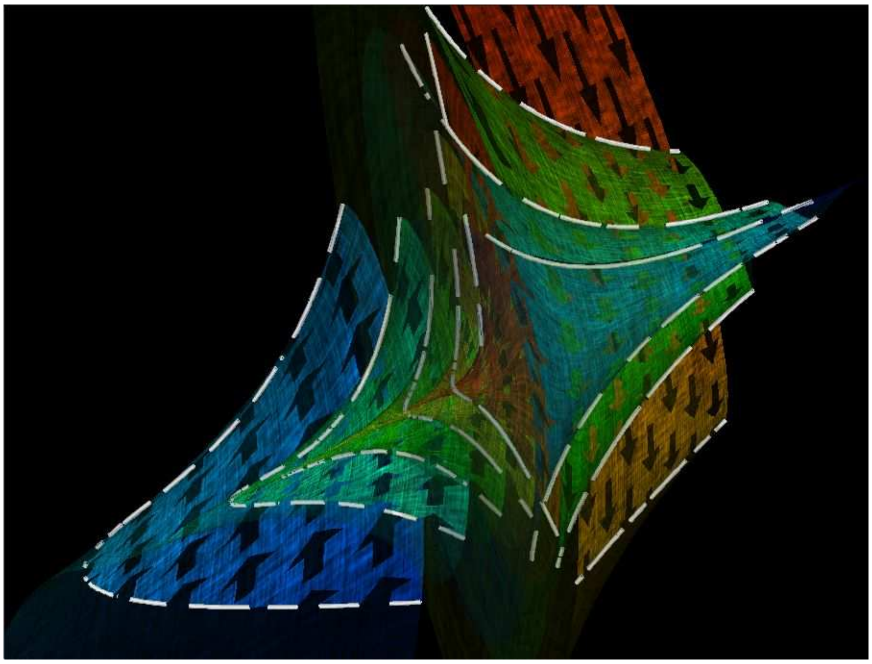

Fig. 11. Streamarrows, anisotropic spot noise, selective cuts