Advanced Modeling 2

Katja Bühler, Andrej Varchola, Eduard Gröller

Institute of Computer Graphics and Algorithms

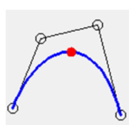
Vienna University of Technology

Topics

- Parametric curves and surfaces
- Polynomial curves
- Rational curves
- Tensor product surfaces
- Triangular surfaces

Parametric Curves

$$c: \quad c(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}; \quad u \in [a,b] =: I \subset IR$$



- $x(u), y(u), z(u)$ are differentiable functions in u
- Tangent vector : $t(u) = \frac{d}{du} c(u)$
- c regular in $c(u_0)$ $\Leftrightarrow$ $t(u_0) \neq o$
- c regular $\Leftrightarrow$ c can be parametrized in a way that all curve points are regular.

Parametric Surfaces

$$s: \quad \mathbf{s}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}; \quad (u,v) \in [a,b] \times [c,d] =: I \times J \subset \mathrm{IR}^2$$

- $x(u,v), y(u,v), z(u,v)$ are differentiable functions in u and v
- Tangent plane : $\mathbf{t}_0(l,m) = \mathbf{s}(u_0,v_0) + l\,\mathbf{s}_u(u_0,v_0) + m\,\mathbf{s}_v(u_0,v_0)$
- Normal vector : $\mathbf{n}(u_0,v_0) = \mathbf{s}_u(u_0,v_0) \times \mathbf{s}_v(u_0,v_0)$
- s regular in $\mathbf{s}(u_0,v_0)$ $\Leftrightarrow$ $\mathbf{n}(u_0,v_0) \neq \mathbf{o}$
- s regular $\Leftrightarrow$ s can be parametrized in a way that all surface points are regular.

Bézier Curves: The de Casteljau Algorithm

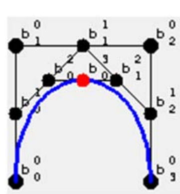
Given n+1 points $\mathbf{b}_0,\dots,\mathbf{b}_n \in IE^3$ and an arbitrary $t \in IR$.
Set

$$\mathbf{b}_i^r := (1\text{-}t)\,\mathbf{b}_i^{r-1} + t\,\mathbf{b}_{i+1}^{r-1} \quad \text{with} \quad \mathbf{b}_i^0 := \mathbf{b}_i.$$

Then $\mathbf{b}_0^n$ is a curve point on the corresponding Bézier curve.
The points $\mathbf{b}_0^{n-1}$ and $\mathbf{b}_1^{n-1}$ determine the tangent line of the curve at point $\mathbf{b}_n^n$.

$$\begin{matrix} \mathbf{b}_0^0 & & & \\ & \mathbf{b}_0^1 & & \\ \mathbf{b}_1^0 & & \mathbf{b}_0^2 & \\ & \mathbf{b}_1^1 & & \mathbf{b}_0^3 \\ \mathbf{b}_2^0 & & \mathbf{b}_1^2 & \\ & \mathbf{b}_2^1 & & \\ \mathbf{b}_3^0 & & & \end{matrix}$$

The de Casteljau scheme



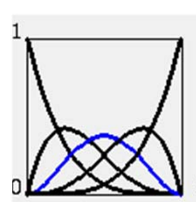
Bézier Curves and Bernstein Polynomials

Bézier curve with respect to Bézier points $b_i,\ i = 0,\dots,n$:

$$b(t) = \sum_{i=0}^{n} b_i B_i^n(t)$$

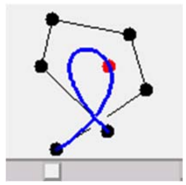
Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$$



Properties of Bézier Curves

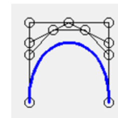
- Affine invariance
- Convex hull property
- Endpoint interpolation
- Linear precision
- Variation diminishing property



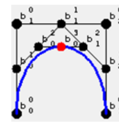
- Disadvantages
  - Only pseudo local control
  - High degree

Important Algorithms for Bézier Curves

- Degree elevation
  - to increase flexibility



- Subdivision
  - to increase flexibility
  - to approximate the curve



The subdivision is a byproduct of the de Casteljau algorithm

Evaluation/Approximation of Bézier Curves

- The **de Casteljau** algorithm is numerical stabil, but inefficient for evaluation
- **Horner scheme** like evaluation is more efficient

$$\mathbf{b}(t) = (...((\binom{n}{0} s\mathbf{b}_0 + \binom{n}{1} t\mathbf{b}_1)s + \binom{n}{2} t^2\mathbf{b}_2)s + ...)s + \binom{n}{n} t^n \mathbf{b}_n \quad \text{with} \quad s = 1\text{-}t$$

- **Repeated subdivision** gives in a fast way a good approximation of the curve

B-Spline Curves

- B-Spline curves
- are piecewise polynomial curves of degree k-1
- have a degree (almost) independent of the number of control points
- allow local control over the shape of a curve

B-Spline Curves: Definition

Given :

- n+1 control points $\mathbf{d}_i \in IE^3,\ i = 0,...,n$
- knot vector $U = (u_0,...,u_{n+k})$

B-spline curve :

$$\mathbf{s}(u) = \sum_{i=0}^{n} \mathbf{d}_i N_i^k(u), \quad u \in [u_0, u_{n+k})$$

with the B-spline basis functions $N_i^k(u)$ of order k.

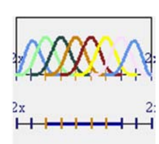
B-Spline Basis Functions

$$k = 0: \quad N_i^0(u) = \begin{cases} 1 & u \in [u_i, u_{i+1}) \\ 0 & sonst \end{cases}$$

$$k > 0: \quad N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u)$$

Properties of the basis functions :

- partition of unity : $\sum_{i=0}^{n} N_i^k(u) \equiv 1$
- positivity : $N_i^k(u) \geq 0$
- local support : $N_i^k(u) = 0$ if $u \notin [u_i, u_{i+k})$



Properties of B-Spline Curves

- Affine invariance
- Strong convex hull property
- Variation diminishing property
- Local support
- Knot points of multiplicity k are coincident with one of the control points.
- A B-Spline curve of order k which has only knots of multiplicity k is a Bézier curve

Evaluating B-Spline Curves: The de Boor Algorithm

Given

n+1 control points $\mathbf{d}_0,\ldots,\mathbf{d}_n \in I\!E^3$, a knot vector $U = (u_0,\ldots,u_{n+k})$ and an arbitrary $t \in [u_0, u_{n+k})$.

Set

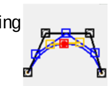
$$\mathbf{d}_i^r := (1-\alpha_i^r)\,\mathbf{d}_{i-1}^{r-1} + \alpha_i^r\,\mathbf{d}_i^{r-1}$$

with $\mathbf{d}_i^0 := \mathbf{d}_i$ and $\alpha_i^r := \dfrac{t-u_i}{u_{i+k-r}-u_i}$

$\mathbf{d}_0^0$
$\mathbf{d}_1^1$
$\mathbf{d}_1^0$ $\mathbf{d}_2^2$
$\mathbf{d}_2^1$ $\mathbf{d}_3^3$
$\mathbf{d}_2^0$ $\mathbf{d}_3^2$
$\mathbf{d}_3^1$
$\mathbf{d}_3^0$

The de Boor scheme

Then $\mathbf{d}_m^{k-1}$ is the point for $\mathbf{x}(t_0)$, $t_0 \in [t_m, t_{m+1}]$ on the corresponding B-spline curve.



The points $\mathbf{d}_{m-1}^{k-2}$ and $\mathbf{d}_m^{k-2}$ determine the tangent line of the curve at point $\mathbf{b}_m^{k-1}$.

Direct Evaluation of B-Spline Curves

- Find the knot span $[u_i,u_{i+1})$ in which the parameter value t lies
- Compute all non zero basis functions
- Multiply the values of the nonzero basis functions with the corresponding control points

Special B-Spline Curves

- open:

$$u_0 = \ldots = u_k < u_{k+1} < \ldots < u_n = \ldots = u_{n+k}$$

- closed:

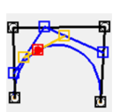
$$\mathbf{d}_{n+1} := \mathbf{d}_0, \ldots, \mathbf{d}_{n+k} := \mathbf{d}_{k-1} \text{ and}$$
$$U = (u_0, \ldots, u_{n+k}, \ldots, u_{n+2k-2})$$

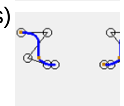
- uniform:

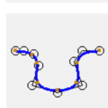
$$U = (u_0, u_0 + d, \ldots, u_0 + (n+k)d)$$

Important Algorithms for B-Spline Curves

- Knot insertion
  - to increase flexibility
  - to compute derivatives
  - to split curves (subdivision algorithms)
  - to evaluate the curve (see de Boor algorithm)
  - to approximate the curve
- Degree elevation
  - to adapt curve degrees







Rational Curves

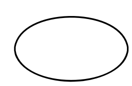
Rational curve :

$$\mathbf{c}(u) = \frac{1}{w(u)} \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}, \quad u \in I \subset I\!R$$

Homogeneous representation :

$$\mathbf{c}_H(u) = \begin{pmatrix} w(u) \\ x(u) \\ y(u) \\ z(u) \end{pmatrix}, \quad u \in I \subset I\!R$$

Example : conic section



$$\mathbf{c}(u) = \frac{1}{1+u^2} \begin{pmatrix} a\,(1-u^2) \\ b\,2u \end{pmatrix}$$

$$\mathbf{c}_H(u) = \begin{pmatrix} 1+u^2 \\ a\,(1-u^2) \\ b\,2u \end{pmatrix}$$

Rational Bézier Curves

A rational Bézier curve is defined as

$$\mathbf{b}(u) = \frac{\sum_{i=0}^{n} w_i \mathbf{b}_i B_i^n(u)}{\sum_{i=0}^{n} w_i B_i^n(u)}, \quad u \in I \subset IR$$

The $w_i > 0,\ i = 0,\ldots,n$ are called weights.

Homogeneous representation:

$$\mathbf{b}_H(u) = \sum_{i=0}^{n} \mathbf{b}_{H\,i}\, B_i^n(u), \quad u \in I \subset IR$$

with the homogeneous Bézier points

$$\mathbf{b}_{H\,i} = \begin{pmatrix} w_i \\ w_i \mathbf{b}_i \end{pmatrix}$$

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Rational Bézier Curves: Properties and Algorithms

- Properties:
  - the same properties like polynomial curve, and
  - projective invariance
  - the weights are an additional design parameter
- Algorithms
  - All algorithms of polynomial Bézier curves can be applied without any change to the homogeneous representation of rational Bézier curves.

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Non Uniform Rational B-Splines

A NURBS curve with respect to the control points $d_i,\ i = 0,\ldots,n$ and the knot vector $U = (u_0,\ldots,u_{n+k})$ is defined as

$$\mathbf{n}(u) = \frac{\sum_{i=0}^{n} w_i \mathbf{d}_i N_i^k(u)}{\sum_{i=0}^{n} w_i N_i^k(u)}, \quad u \in [u_0, u_{n+k}) \subset IR$$

The $w_i > 0,\ i = 0,\ldots,n$ are called weights.

Homogeneous representation:

$$\mathbf{n}_H(u) = \sum_{i=0}^{n} \mathbf{n}_{H\,i}\, N_i^k(u), \quad u \in [u_0, u_{n+k}) \subset IR$$

with the homogeneous B-Spline points

$$\mathbf{n}_{H\,i} = \begin{pmatrix} w_i \\ w_i \mathbf{n}_i \end{pmatrix}$$

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NURBS: Properties and Algorithms

- Properties:
  - the same properties like polynomial curve, and
  - projective invariance
  - changing the weight w_i affects only the interval $[u_i, u_{i+k})$
- Algorithms
  - All algorithms of B-spline curves can be applied without any change to the homogeneous representation of NURBS curves.

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Tensor-Product Surfaces

- "A surface is the locus of a curve that is moving through space and thereby changing the shape"

Given a curve

$$\mathbf{f}(u) = \sum_{i=0}^{n} \mathbf{c}_i F_i(u), \quad u \in I \subset IR$$

moving the control points yields

$$\mathbf{c}_i(v) = \sum_{j=0}^{m} \mathbf{a}_{ij} G_j(v), \quad v \in J \subset IR$$

Combining both yields a tensor-product surface

$$s(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{a}_{ij} F_i(u) G_j(v), \quad (u,v) \in I \times J \subset IR^2$$

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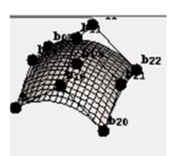
Tensor-Product Bézier Surfaces

A tensor-product Bézier surface is given by

$$\mathbf{b}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v), \quad (u,v) \in I \times J \subset IR^2$$

The Bézier points $\mathbf{b}_{ij}$ form the control net of the surface.



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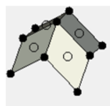
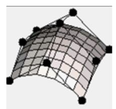
Tensor-Product Bézier Surfaces

- Properties:
  analogue to that of Bézier curves
- Algorithms:
  Apply algorithms for curves in two steps:
  - Apply to
    $$\mathbf{b}_i(v) = \sum_{j=0}^{m} \mathbf{b}_{ij} B_j^m(v), \quad i = 0,\dots,n$$
  - Apply to
    $$\mathbf{b}(u,v) = \sum_{i=0}^{n} \mathbf{b}_i(v) B_i^n(u)$$

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Tensor-Product B-Spline Surfaces

A tensorproduct B-spline surface with respect to the knot vectors
$U = (u_0,\dots,u_{n+k}), V = (v_0,\dots,v_{m+l})$
is given by

$$\mathbf{d}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{d}_{ij} N_i^k(u) N_j^l(v), \quad (u,v) \in [u_0,u_{n+k}) \times [v_0,v_{m+l}) \subset IR^2.$$

The control points $\mathbf{d}_{ij}$ form the control net of the surface.

Properties and Algorithms are analogue to the description for Bézier tensor product surfaces.

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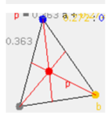
Bézier Triangles

A triangular Bézier patch is defined by

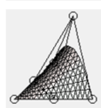
$$b(u,v,w) = \sum_{\substack{i+j+k=n \\ i,j,k>0}} \mathbf{b}_{ijk} B_{ijk}^n(u,v,w)$$

The u,v,w are barycentric coordinates of the triangular parameter domain.



Generalized Bernstein polynomials:

$$B_{ijk}^n(u,v,w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k$$



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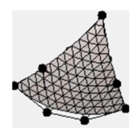
Bézier Triangles: Properties and Algorithms

- Properties:
  - the same as in the univariate case
- Algorithms:
  - De Casteljau:
    $$b_{ijk}^l = u b_{i+1jk}^{l-1} + v b_{ij+1k}^{l-1} + w b_{ijk+1}^{l-1}$$
  - Subdivision
  - Degree elevation

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Subdivision Surfaces

- Polygon-mesh surfaces generated from a base mesh through an iterative process that smoothes the mesh while increasing its density
- Represented as functions defined on a parametric domain with values in R^3
- Allow to use the initial control mesh as the domain
- Developed for the purpose of CG and animation

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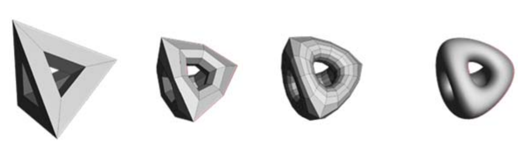
Subdivision Surfaces: The Basic Idea

- In each iteration
  - Refine the initial control mesh
  - Increase the number of vertices / faces
- The mesh vertices converge to a limit surface



SIGGRAPH 2000 Course Notes

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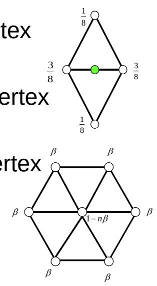
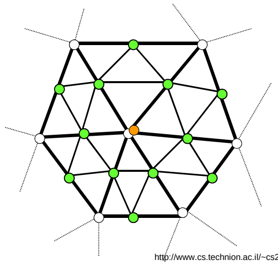
Loop's Scheme ('87)

- Old vertex
- Edge vertex
- New vertex





$$\beta = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{2}{8}\cos\left(\frac{2\pi}{n}\right)\right)^2\right)$$

Catmull-Clark Scheme '78

- Face vertex

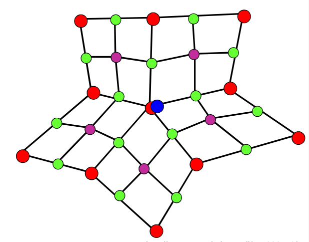
$$v_f = \frac{1}{4}\sum_{i=1}^{4} v_i$$

- Edge vertex

$$v_e = \frac{v_1 + v_2 + v_{f1} + v_{f2}}{4}$$

- Vertex

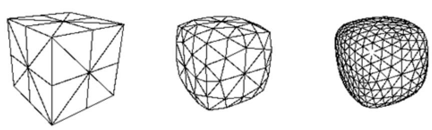
$$v = \frac{Q}{n} + \frac{2R}{n} + \frac{p\,(n-3)}{n}$$



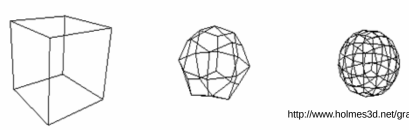
- Q – Average of face vertices
- R – Average of edge vertices
- v – new vertex

Comparison of the Loop and the Catmull-Clark Scheme

Loop subdivision scheme:



Catmull-Clark subdivision scheme:



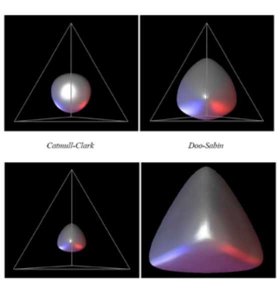
http://www.holmes3d.net/graphics/subdivision/

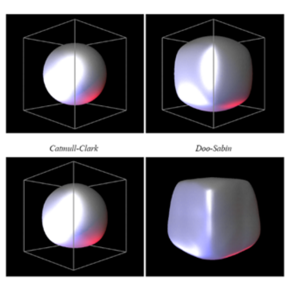
Andrej Va

Subdivision Surfaces: Classification

- The type of refinement rule
  - Vertex insertion
  - Corner cutting
- The type of generated mesh
  - Triangular
  - Quadrilateral
- Approximating vs. Interpolating

Subdivision Surfaces: Comparison





[http://www.cs.technion.ac.il/~cs236716/]