# Sampling and Reconstruction

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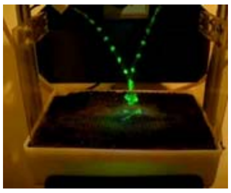
## Motivation

- Theory and practice of sampling and reconstruction
- Aliasing
  - Understanding the problem
  - Handling the problem
- Examples:
  - Photography/Video
  - Rendering
  - Computed Tomography
  - Collision detection



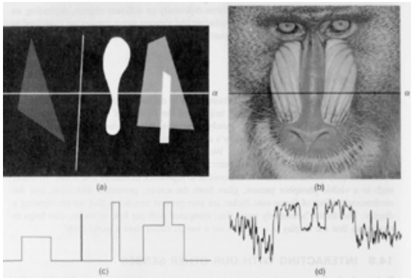


## Overview

- Introduction
- Tools for Sampling and Reconstruction
  - Fourier Transform
  - Convolution (dt.: Faltung)
  - Convolution Theorem
  - Filtering
- Sampling
  - The mathematical model
- Reconstruction
  - Sampling Theorem
  - Reconstruction in Practice



## Image Data

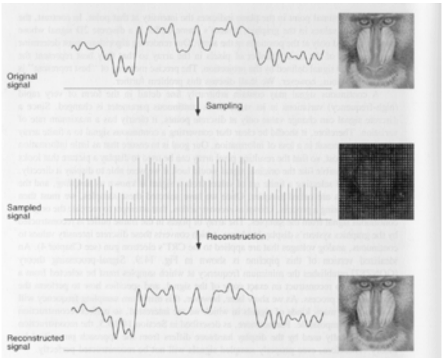



**Demo** - Scan Lines
- Sampling

http://www.cs.brown.edu/exploratories/freeSoftware/repository/edu/brown/cs/exploratories/applets/sampling/introduction_to_sampling_java_browser.html

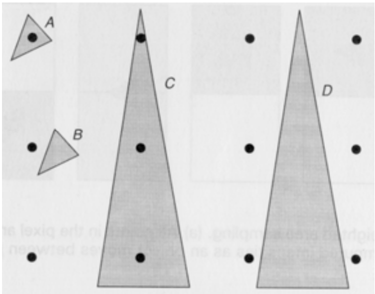


## Image Storage and Retrieval






## Sampling Problems



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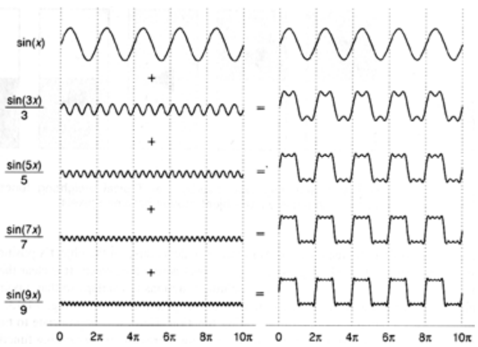
## Sampling Theory

- Relationship between signal and samples
- View image data as signals
- Signals are plotted as intensity vs. spatial domain
- Signals are represented as sum of sine waves - frequency domain



## Square Wave Approximation






## Fourier Series

- Eq1: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(n\omega x - \varphi_n)\right]$$
- Eq2: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega x) + b_n \sin(n\omega x)\right]$$
- Eq3: $$f(x) = \sum_{n=-\infty}^{\infty}\left[c_n e^{in\omega x}\right]$$
- Euler's identity: $e^{ix} = \cos x + i \sin x$



## Fourier Transform

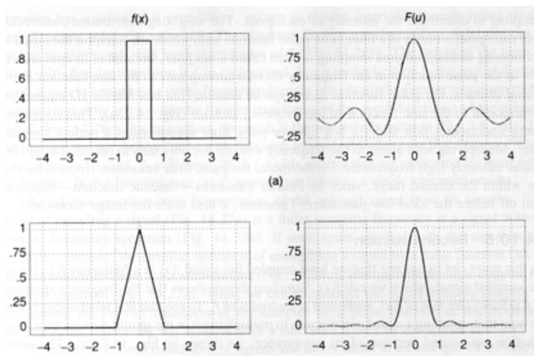
- Link between spatial and frequency domain

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx$$

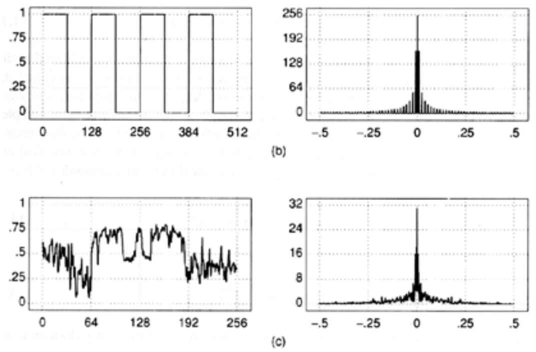
$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$



## Box & Tent



## Square Wave & Scanline






## Fourier Transform

- Yields complex functions for frequency domain
- Extends to higher dimensions
- Complex part is phase information - usually ignored in explanation and visual analysis

- Alternative:
  - Hartley transform
  - Wavelet transform



## Discrete Fourier Transform

- For discrete signals (i.e., sets of samples)

$$f(x) = \sum_{\omega=0}^{N-1} F(\omega) \cdot e^{2\pi j \omega x / N}$$

$$F(\omega) = \frac{1}{N} \sum_{\omega=0}^{N-1} f(x) \cdot e^{-2\pi j \omega x / N}$$

- Complexity for N samples:
  - DFT $O(N^2)$
  - Fast FT (FFT): O(N log N)

**Demo** - Fast Fourier Transform
- Different Frequencies

http://www.cs.brown.edu/exploratories/freeSoftware/repository/edu/brown/cs/exploratories/applets/fft1DApp/1d_fast_fourier_transform_java_browser.html



## Overview





## Convolution

- Operation on two functions
- Result: Sliding weighted average of a function
- The second function provides the weights

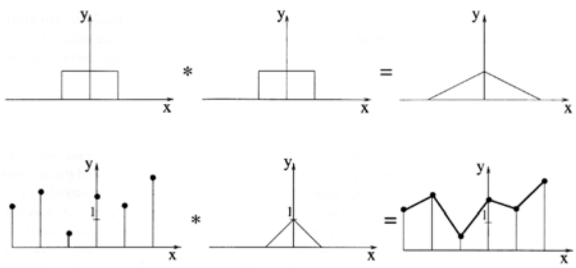
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\tau) g(\tau) d\tau$$

**Demo** - Copying (Dirac Pulse)
- Averaging (Box Filter)

http://www.cs.brown.edu/exploratories/freeSoftware/repository/edu/brown/cs/exploratories/applets/convolution/convolution_java_browser.html



## Convolution - Examples



## Convolution Theorem

- The spectrum of the convolution of two functions is equivalent to the product of the transforms of both input signals, and vice versa.

$$f_1 * f_2 \equiv F_1 F_2$$

$$F_1 * F_2 \equiv f_1 f_2$$

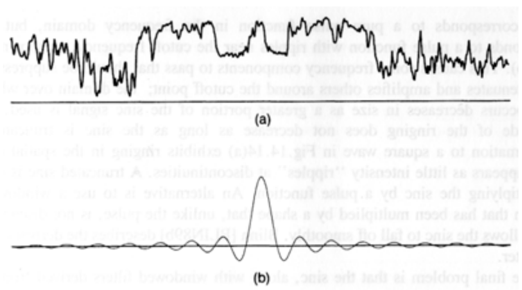


## Example - Low-Pass

- Low-pass filtering performed on Mandrill scanline
- **Spatial domain**: convolution with sinc function
- **Frequency domain**: cutoff of high frequencies - multiplication with box filter
- *Sinc function corresponds to box function and vice versa!*

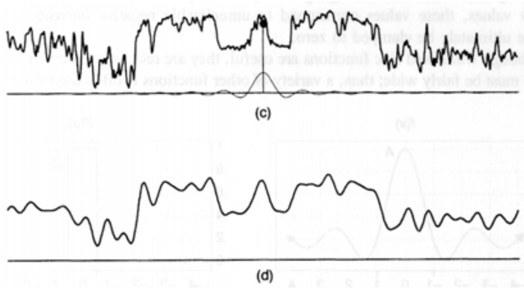


## Low-Pass in Spatial Domain 1






## Low-Pass in Spatial Domain 2



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## Sampling

- The process of sampling is a multiplication of the signal with a comb function

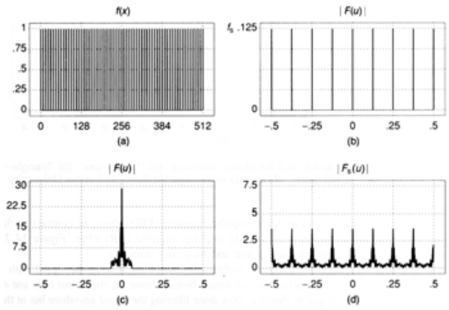
$$f_s(x) = f(x) \cdot \mathrm{comb}_T(x)$$

- The frequency response is convolved with a transformed comb function.

$$F_s(\omega) = F(\omega) * \mathrm{comb}_{1/T}(\omega)$$

## FT of Base Functions

- Comb function: $\text{comb}_T(x) \Leftrightarrow \text{comb}_{1/T}(\omega)$



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## Reconstruction

*Recovering the original function*
*from a set of samples*

- Sampling theorem
- Ideal reconstruction
  - Sinc function
- Reconstruction in practice



## Definitions

- A function is called *band-limited* if it contains no frequencies outside the interval [-u,u]. u is called the *bandwidth* of the function
- The *Nyquist frequency* of a function is twice its bandwidth, i.e. w = 2u



## Sampling Theorem

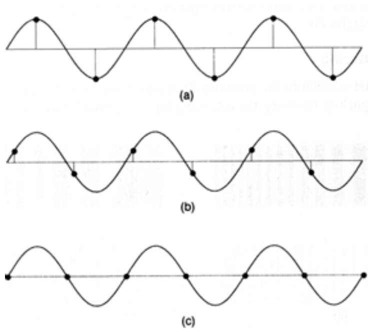
A function f(x) that is

- band-limited and
- sampled above the Nyquist frequency

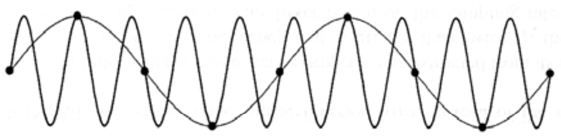
is completely determined by its samples.



## Sampling at Nyquist Frequency



## Sampling Below Nyquist Frequency



**Demo**
- Under sampling
- Nyquist Frequency

http://www.cs.brown.edu/exploratories/freeSoftware/repository/edu/brown/cs/exploratories/applets/nyquist/nyquist_limit_java_browser.html



## Ideal Reconstruction

- Replicas in frequency domain must not overlap
- Multiplying the frequency response with a box filter of the width of the original bandwidth restores original
- Amounts to convolution with Sinc function



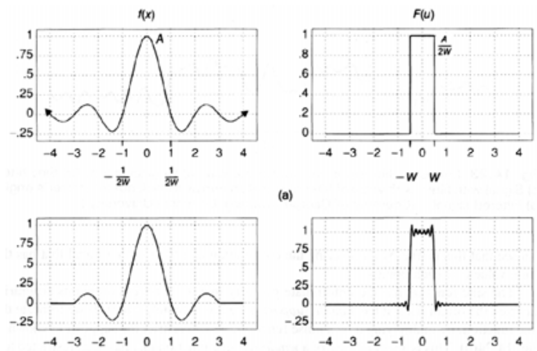
## Sinc Function

- Infinite in extent
- Ideal reconstruction filter
- FT of box function

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



## Sinc & Truncated Sinc



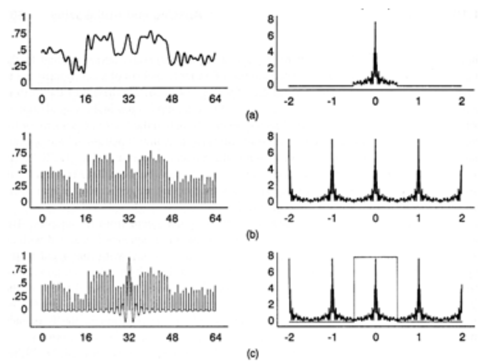



## Reconstruction: Examples

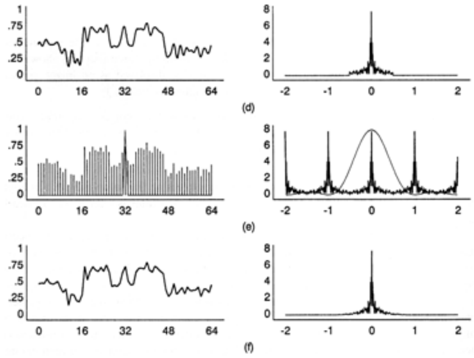
- Sampling and reconstruction of the Mandrill image scanline signal
- with adequate sampling rate
- with inadequate sampling rate
- demonstration of band-limiting
- With Sinc and tent reconstruction kernels



## Adequate Sampling Rate

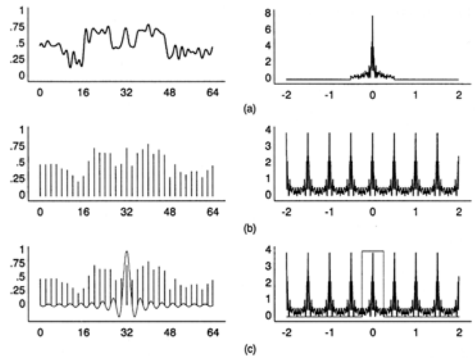


## Adequate Sampling Rate



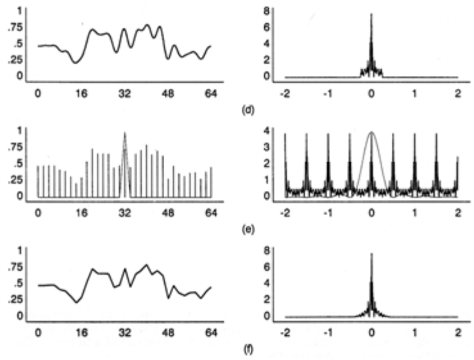



## Inadequate Sampling Rate



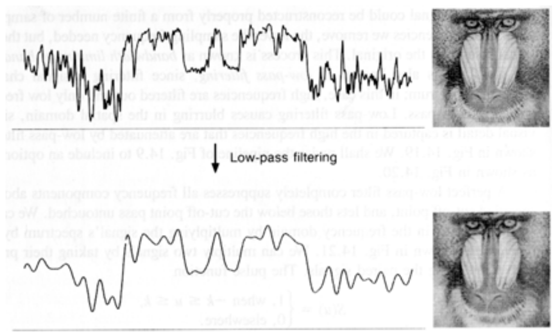



## Inadequate Sampling Rate






## Band-Limiting a Signal



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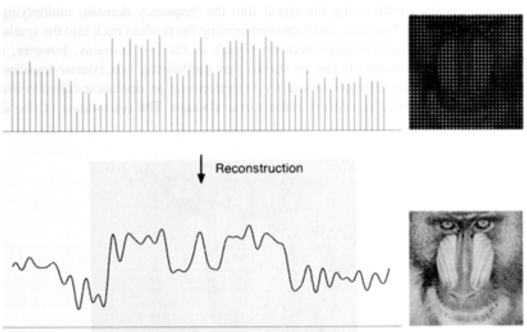





## Reconstruction in Practice

- Problem: which reconstruction kernel should be used?
  - Genuine Sinc function unusable in practice
  - Truncated Sinc often sub-optimal
- Various approximations exist; none is optimal for all purposes

## Tasks of Reconstruction Filters

- Remove the extraneous replicas of the frequency response
- Retain the original undistorted frequency response



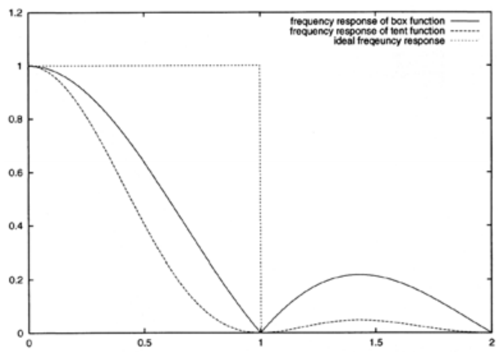
## Used Reconstruction Filters

- Nearest neighbour
- Linear interpolation
- Symmetric cubic filters
- Windowed Sinc

  More sophisticated ways of truncating the Sinc function

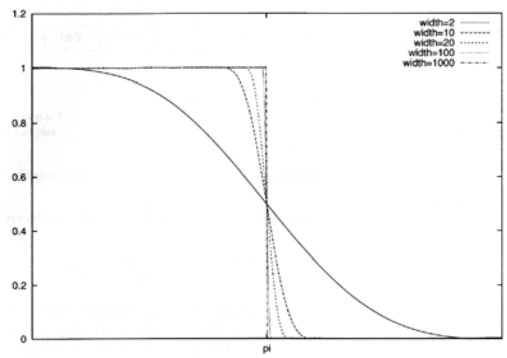


## Box & Tent Responses






## Windowed Sinc Responses






## Sampling & Reconstruction Errors

- Aliasing: due to overlap of original frequency response with replicas - information loss
- Truncation Error: due to use of a finite reconstruction filter instead of the infinite Sinc filter
- Non-Sinc error: due to use of a reconstruction filter that has a shape different from the Sinc filter



## Interpolation - Zero Insertion

- Operates on series of *n* samples
- Takes advantage of DFT properties

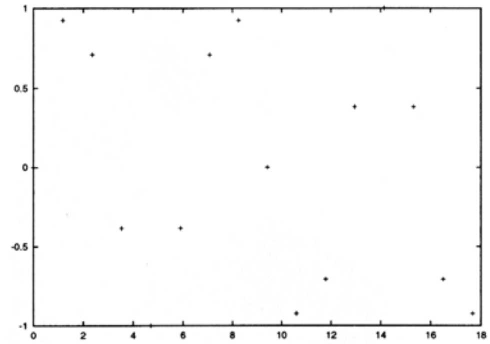
- Algorithm:
  - Perform DFT on series
  - Append zeros to the sequence
  - Perform the inverse DFT

## Zero Insertion - Properties

- Preserves frequency spectrum
- Original signal has to be sampled above Nyquist frequency
- Values can only be interpolated at evenly spaced locations
- The whole series must be accessible, and it is always completely processed

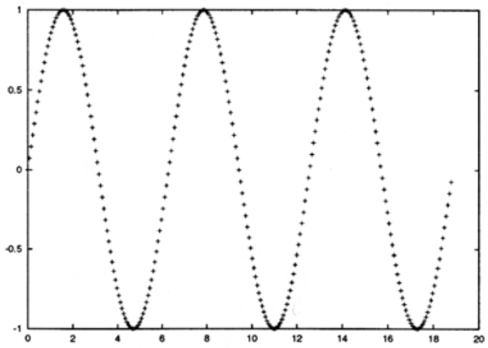


## Zero Insertion - Original Series






## Zero Insertion - Interpolation






## Conclusion

- Sampling
  - Going from continuous to discrete signal
  - Mathematically modeled with a multiplication with comb function
  - Sampling theorem: How many samples are needed
- Reconstruction
  - Sinc is the ideal filter but not practicable
  - Reconstruction in practice
  - Aliasing



## Sampling and Reconstruction

- References:
  - Computer Graphics: Principles and Practice, 2nd Edition, Foley, vanDam, Feiner, Hughes, Addison-Wesley, 1990
  - What we need around here is more aliasing, Jim Blinn, IEEE Computer Graphics and Applications, January 1989
  - Return of the Jaggy, Jim Blinn, IEEE Computer Graphics and Applications, March 1989
  - Demo Applets, Brown University, Rhode Island, USA http://www.cs.brown.edu/exploratories/freeSoftware/home.html



## Questions