Sampling and Reconstruction

Overview

- Introduction
- Sampling Theory
  - Fourier Transform
  - Convolution & Convolution Theorem
- Reconstruction
  - Sampling Theorem
  - Reconstruction in theory and practice
- Interpolation - Zero Insertion

Image Data

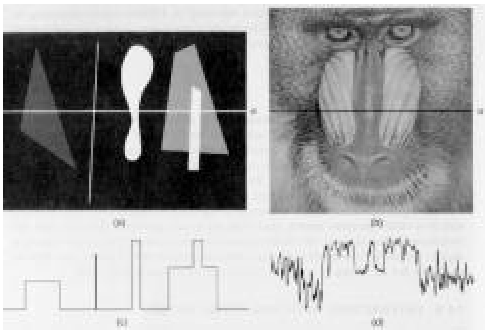
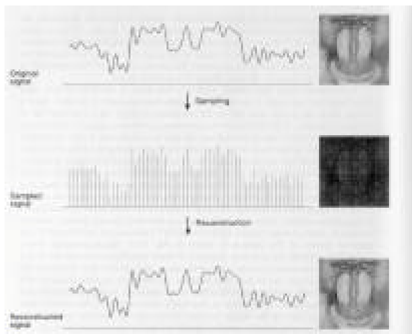
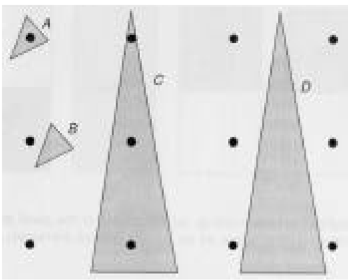


Image Storage and Retrieval



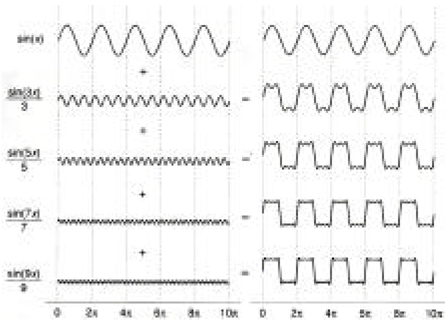
Sampling Problems



Sampling Theory

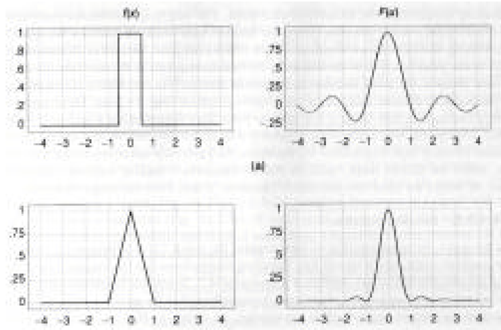
- Relationship between Signal and Samples
- View Image Data as Signals
- Signals can be plotted as intensity vs. time - spatial domain
- Signals can be represented as sum of sine waves - frequency domain

Square Wave Approximation



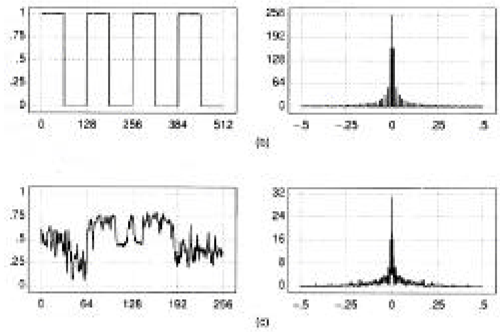
Eduard Gröller, Thomas Theußl 7 /49

Box & Tent



Eduard Gröller, Thomas Theußl 8 /49

Square Wave & Scanline



Eduard Gröller, Thomas Theußl 9 /49

Fourier Transform

Link between spatial and frequency domain

$$\mathrm{f}(x) = \int_{-\infty}^{\infty} \mathrm{F}(?)e^{2\mathrm{pj}?x}dw$$

$$\mathrm{F}(?) = \int_{-\infty}^{\infty} \mathrm{f}(x)e^{-2\mathrm{pj}?x}dx$$

Eduard Gröller, Thomas Theußl 10 /49

Fourier Transform

- Yields complex functions for frequency domain
- Extends to higher dimensions
- Complex part is phase information - usually ignored

Alternative: Hartley transform

Eduard Gröller, Thomas Theußl 11 /49

Discrete Fourier Transform

For discrete signals (i.e. sets of samples)

$$\mathrm{f}(x) = \sum_{?=0}^{\mathrm{N-1}} \mathrm{F}(?) \cdot e^{2pjwx/\mathrm{N}}$$

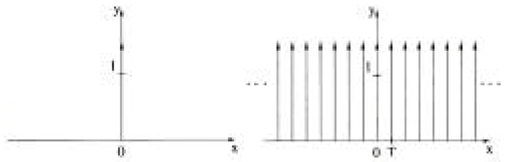
$$\mathrm{F}(?) = \frac{1}{\mathrm{N}} \sum_{?=0}^{\mathrm{N-1}} \mathrm{f(x)} \cdot e^{-2pjwx/\mathrm{N}}$$

N samples: O(N^2) complexity

Fast FT (FFT): O(N log N)

Eduard Gröller, Thomas Theußl 12 /49

Base Functions



Dirac Pulse   Comb Function

FT of Base Functions

- Impulse function: constant 1, i.e. equal energy at all frequencies
- Comb function: comb with reciprocal spacing

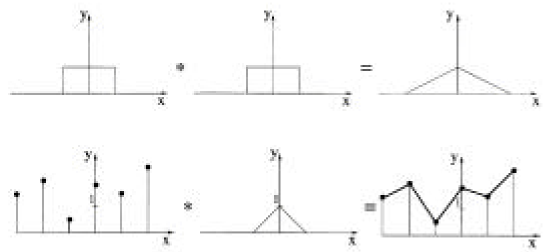
$$\mathrm{comb}_T(x) \Leftrightarrow \mathrm{comb}_{1/T}(?)$$

Convolution

- Operation on two functions
- Produces a new function which is a sliding weigthed average of a function. The second function provides the weights.

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(x_1^{'}) f_2(x - x^{'}) dx^{'}$$

Convolution - Examples



Convolution Theorem

The spectrum of the convolution of two functions is equivalent to the product of the transforms of both input signals, and vice versa.

$$f_1 * f_2 \equiv F_1 F_2$$
$$F_1 * F_2 \equiv f_1 f_2$$

Example - Low-Pass

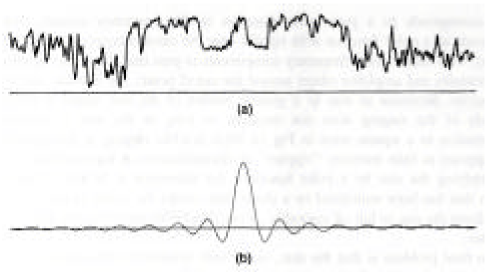
Low-pass filtering performed on Mandrill scanline

Spatial domain: convolution with sinc function

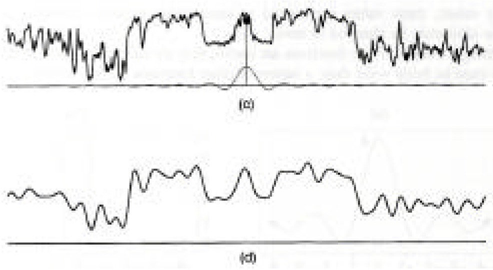
Frequency domain: cutoff of high frequencies - multiplication with box filter

Sinc function corresponds to box function and vice versa!

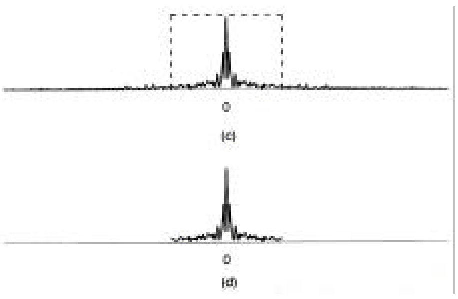
Low-Pass in Spatial Domain 1



Low-Pass in Spatial Domain 2



Low-Pass in Frequency Domain



Sampling

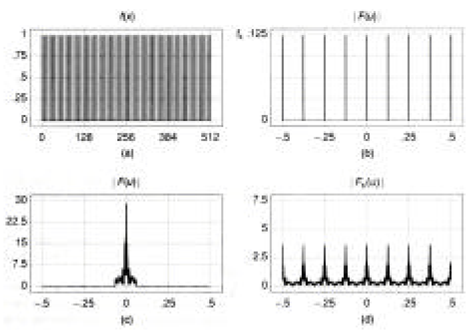
The process of sampling is a multiplication of the signal with a comb function.

$$f_s(x) = f(x) \cdot \text{comb}_T(x)$$

The frequency response is convolved with a transformed comb function.

$$F_s(w) = F(w) * \text{comb}_{1/T}(w)$$

Spectrum of a Sampled Signal



Reconstruction

Recovering the original function from a set of samples

- Sampling theorem
- Ideal reconstruction
  - Sinc function
- Reconstruction in practice

Definitions

- A function is called *band-limited* if it contains no frequencies outside the interval [-u,u]. u is called the *bandwidth* of the function
- The *Nyquist frequency* of a function is twice its bandwidth, i.e. w = 2u

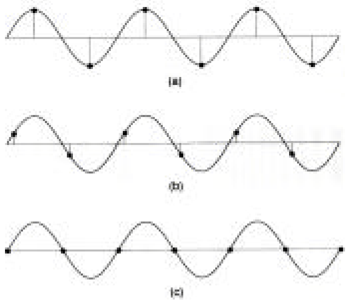
Sampling Theorem

A function f(x) that is

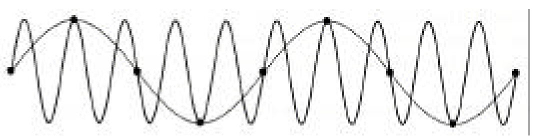
- band-limited and
- sampled above the Nyquist frequency

is completely determined by its samples.

Sampling at Nyquist Frequency



Sampling Below Nyquist *f*



Ideal Reconstruction

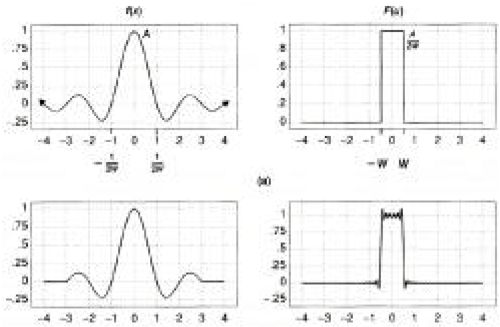
- Replicas in frequency domain must not overlap
- Multiplying the frequency response with a box filter of the width of the original bandwidth restores original
- Amounts to convolution with Sinc function

Sinc function

- Infinite in extent
- Ideal reconstruction filter
- FT of box function

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin px}{px} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$$

Sinc & Truncated Sinc



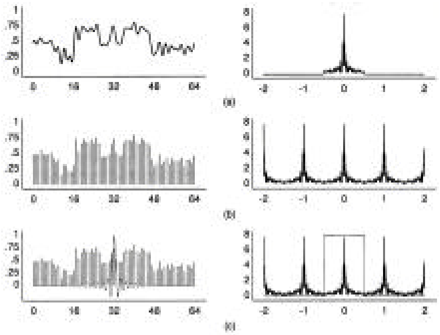
Reconstruction: Examples

Sampling and reconstruction of the Mandrill image scanline signal

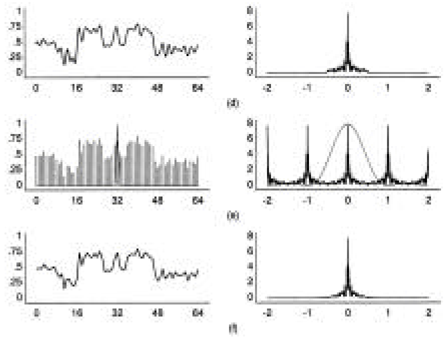
- with adequate sampling rate
- with inadequate sampling rate
- demonstration of band-limiting

With Sinc and tent reconstruction kernels

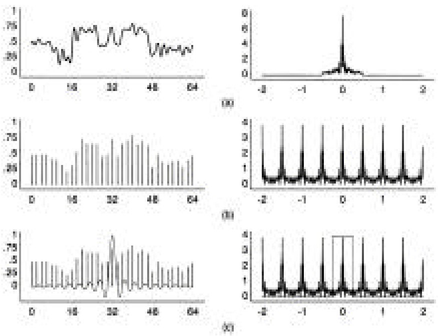
Adequate Sampling Rate



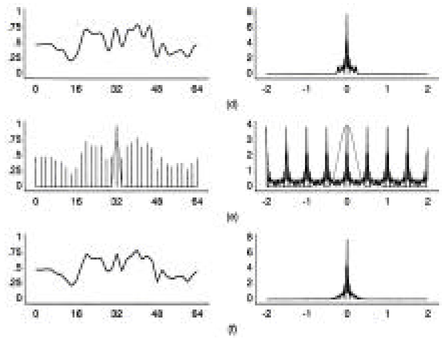
Adequate Sampling Rate



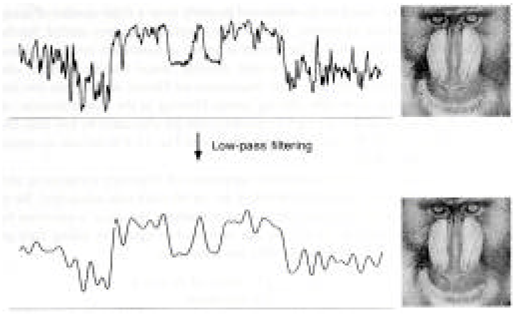
Inadequate Sampling Rate



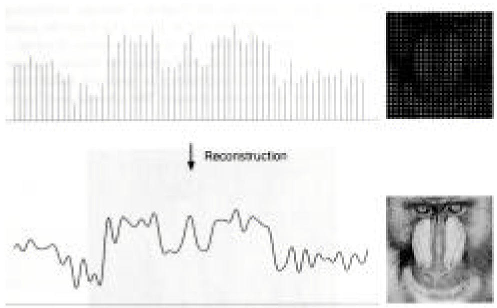
Inadequate Sampling Rate



Band-Limiting a Signal



Band-Limiting a Signal



Reconstruction in Practice

Problem: which reconstruction kernel should be used?

- Genuine Sinc function unusable in practice
- Truncated Sinc often sub-optimal
- Various approximations exist; none is optimal for all purposes

Tasks of Reconstruction Filters

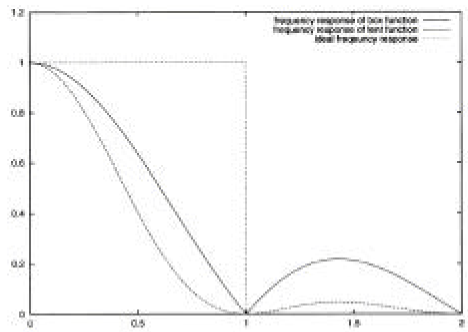
- Remove the extraneous replicas of the frequency response
- Retain the original undistorted frequency response

Used Reconstruction Filters

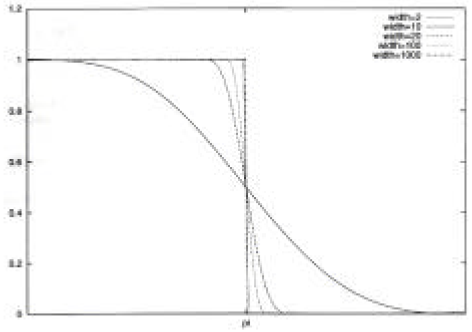
- Nearest neighbour
- Linear interpolation
- Symmetric cubic filters
- Windowed Sinc

  More sophisticated ways of truncating the Sinc function

Box & Tent Responses



Windowed Sinc Responses



Sampling & Reconstruction Errors

- Aliasing: due to overlap of original frequency response with replicas - information loss
- Truncation Error: due to use of a finite reconstruction filter instead of the infinite Sinc filter
- Non-Sinc error: due to use of a reconstruction filter that has a shape different from the Sinc filter

Interpolation - Zero Insertion

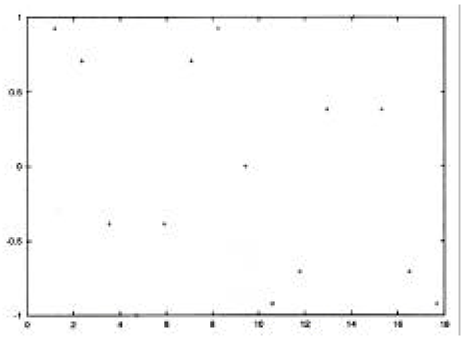
Operates on series of *n* samples
Takes advantage of DFT properties

- Perform DFT on series
- Append zeros to the sequence
- Perform the inverse DFT

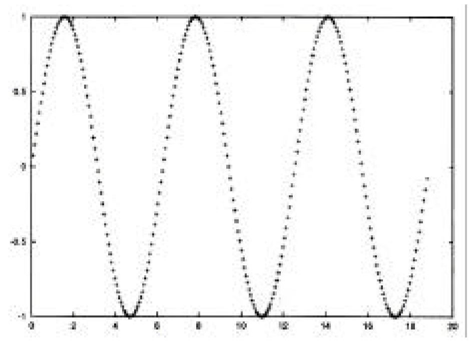
Zero Insertion - Properties

- Preserves frequency spectrum
- Original signal has to be sampled above Nyquist frequency
- Values can only be interpolated at evenly spaced locations
- The whole series must be accessible, and it is always completey processed

Zero Insertion - Original Series



Zero Insertion - Interpolation



Sampling and Reconstruction

References:

- Computer Graphics: Principles and Practice, 2nd Edition, Foley, vanDam, Feiner, Hughes, Addison-Wesley, 1990
- What we need around here is more aliasing, Jim Blinn, IEEE Computer Graphics and Applications, January 1989
- Return of the Jaggy, Jim Blinn, IEEE Computer Graphics and Applications, March 1989