A Short Survey on Curves and Surfaces in CAGD

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Topics

Parametric curves and surfaces

Polynomial curves

Rational curves

Tensor product surfaces

Triangular surfaces

Parametric Curves

$$c: \quad \mathbf{c}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}; \quad u \in [a,b] =: I \subset \mathrm{IR}$$

- $x(u), y(u), z(u)$ are differentiable functions in u
- Tangent vector : $\mathbf{t}(u) = \frac{d}{du}\mathbf{c}(u)$
- c regular in $\mathbf{c}(u_0) \Leftrightarrow \mathbf{t}(u_0) \neq \mathbf{o}$
- c regular $\Leftrightarrow$ c can be parametrized in a way that all curve points are regular.

Parametric Surfaces

$$s: \quad \mathbf{s}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}; \quad (u,v) \in [a,b] \times [c,d] =: I \times J \subset \mathrm{IR}^2$$

- $x(u,v), y(u,v), z(u,v)$ are differentiable functions in u and v
- Tangent plane : $\mathbf{t}_0(l,m) = \mathbf{s}(u_0,v_0) + l\,\mathbf{s}_u(u_0,v_0) + m\,\mathbf{s}_v(u_0,v_0)$
- Normal vector : $\mathbf{n}(u_0,v_0) = \mathbf{s}_u(u_0,v_0) \times \mathbf{s}_v(u_0,v_0)$
- s regular in $\mathbf{s}(u_0,v_0) \Leftrightarrow \mathbf{n}(u_0,v_0) \neq \mathbf{o}$
- s regular $\Leftrightarrow$ s can be parametrized in a way that all surface points are regular.

Bézier Curves: The de Casteljau Algorithm

Given n + 1 points $\mathbf{b}_0,...,\mathbf{b}_n \in IE^3$ and an arbitrary $t \in IR$.

Set

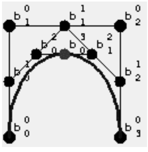
$$\mathbf{b}_i^r := (1\text{-}t)\,\mathbf{b}_i^{r-1} + t\,\mathbf{b}_{i+1}^{r-1} \quad \text{with } \mathbf{b}_i^0 := \mathbf{b}_i.$$

Then $\mathbf{b}_0^n$ is a curve point on the corresponding Bézier curve.

The points $\mathbf{b}_0^{n-1}$ and $\mathbf{b}_1^{n-1}$ determine the tangent line of the curve at point $\mathbf{b}_n^n$.

$\mathbf{b}_0^0$
$\mathbf{b}_0^1$
$\mathbf{b}_1^0$ $\mathbf{b}_0^1$
$\mathbf{b}_1^1$ $\mathbf{b}_0^3$
$\mathbf{b}_2^0$ $\mathbf{b}_1^2$
$\mathbf{b}_2^1$
$\mathbf{b}_3^0$

The de Casteljau scheme



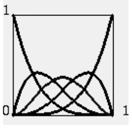
Bézier Curves and Bernstein Polynomials

Bézier curve with respect to Bézier points $\mathbf{b}_i$, i = 0,..., n :

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t)$$

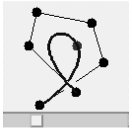
Bernstein polynomials :

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i} t^i$$



Properties of Bézier Curves

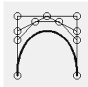
Affine invariance
Convex hull property
Endpoint interpolation
Linear precision
Variation diminishing property



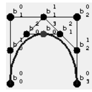
Disadvantages
Only pseudo local control
High degree

Important Algorithms for Bézier Curves

Degree elevation
- to increase flexibility



Subdivision
- to increase flexibility
- to approximate the curve



The subdivision is a byproduct of the de Casteljau algorithm

Evaluation/Approximation of Bézier Curves

The **de Casteljau** algorithm is numerical stabil, but inefficient for evaluation

Horner scheme like evaluation is more efficient

$$\mathbf{b}(t) = (\ldots((\binom{n}{0} s\mathbf{b}_0 + \binom{n}{1} t\mathbf{b}_1)s + \binom{n}{2} t^2\mathbf{b}_2)s + \ldots)s + \binom{n}{n} t^n \mathbf{b}_n \quad \text{with } s = 1\text{-}t$$

Repeated subdivision gives in a fast way a good approximation of the curve

B-Spline Curves

B-spline curves

are piecewise polynomial curves of degree k-1

have a degree (almost) independent of the number of control points

allow local control over the shape of a curve

B-Spline Curves: Definition

Given :

- n+1 control points $\mathbf{d}_i \in IE^3,\ i = 0,\ldots,n$
- knot vector $U = (u_0,\ldots,u_{n+k})$

B-spline curve :

$$\mathbf{s}(u) = \sum_{i=0}^{n} \mathbf{d}_i N_i^k(u), \quad u \in [u_0, u_{n+k})$$

with the B-spline basis functions $N_i^k(u)$ of order k.

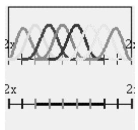
B-Spline Basis Functions

$$k = 1: \quad N_i^1(u) = \begin{cases} 1 & u \in [u_i, u_{i+1}) \\ 0 & sonst \end{cases}$$

$$k > 1: \quad N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i}^{k-1}(u)$$

Properties of the basis functions :

- partition of unity : $\sum_{i=0}^{n} N_i^k(u) \equiv 1$
- positivity : $N_i^k(u) \geq 0$
- local support : $N_i^k(u) = 0 \quad \text{if } u \notin [u_i, u_{i+k})$



Properties of B-Spline Curves

Affine invariance

Strong convex hull property

Variation diminishing property

Local support

Knot points of multiplicity k are coincident with one of the control points.

A B-spline curve of order k which has only knots of multiplicity k is a Bézier curve

Evaluating B-Spline Curves: The de Boor Algorithm

Given

n+1 control points $\mathbf{d}_0,\dots,\mathbf{d}_n \in IE^3$, a knot vector $U = (u_0,\dots,u_{n+k})$ and an arbitrary $t \in [u_0, u_{n+k})$.

Set

$$\mathbf{d}_i^r := (1-\alpha_i^r)\,\mathbf{d}_{i-1}^{r-1} + \alpha_i^r\,\mathbf{d}_i^{r-1}$$

with $\mathbf{d}_i^0 := \mathbf{d}_i$ and $\alpha_i^r := \dfrac{t-u_i}{u_{i+k-r}-u_i}$

Then $\mathbf{d}_m^{k-1}$ is the point for $\mathbf{x}(t_0)$, $t_0 \in [t_m, t_{m+1}]$ on the corresponding B-spline curve.

The points $\mathbf{d}_{m-1}^{k-2}$ and $\mathbf{d}_m^{k-2}$ determine the tangent line of the curve at point $\mathbf{b}_m^{k-1}$.

$\mathbf{d}_0^0$

$\mathbf{d}_1^1$

$\mathbf{d}_1^0$ $\mathbf{d}_2^2$

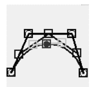
$\mathbf{d}_2^1$ $\mathbf{d}_3^3$

$\mathbf{d}_2^0$ $\mathbf{d}_3^2$

$\mathbf{d}_3^1$

$\mathbf{d}_3^0$

The de Boor scheme



Direct Evaluation of B-Spline Curves

Find the knot span $[u_i,u_{i+1})$ in which the parameter value t lies

Compute all non zero basis functions

Multiply the values of the nonzero basis functions with the corresponding control points

Special B-Spline Curves

open:

$$u_0 = \dots = u_k < u_{k+1} < \dots < u_n = \dots = u_{n+k}$$

closed:

$$\mathbf{d}_{n+1} := \mathbf{d}_0, \dots, \mathbf{d}_{n+k} := \mathbf{d}_{k-1} \text{ und}$$
$$U = (u_0, \dots, u_{n+k}, \dots, u_{n+2k-2})$$

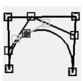
uniform:

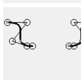
$$U = (u_0, u_0 + d, \dots, u_0 + (n+k)d)$$

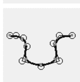
Important Algorithms for B-Spline Curves

Knot insertion
- to increase flexibility
- to compute derivatives
- to split curves (subdivision algorithms)
- to evaluate the curve (see de Boor algorithm)
- to approximate the curve

Degree elevation
- to adapt curve degrees







Rational Curves

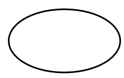
Rational curve:

$$\mathbf{c}(u) = \frac{1}{w(u)}\begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}, \quad u \in I \subset IR$$

Homogeneous representation:

$$\mathbf{c}_H(u) = \begin{pmatrix} w(u) \\ x(u) \\ y(u) \\ z(u) \end{pmatrix}, \quad u \in I \subset IR$$

Example: conic section



$$\mathbf{c}(u) = \frac{1}{1+u^2}\begin{pmatrix} a\,(1-u^2) \\ b\,2u \end{pmatrix}$$

$$\mathbf{c}_H(u) = \begin{pmatrix} 1+u^2 \\ a\,(1-u^2) \\ b\,2u \end{pmatrix}$$

Rational Bézier Curves

A rational Bézier curve is defined as

$$\mathbf{b}(u) = \frac{\sum_{i=0}^{n} w_i \mathbf{b}_i B_i^n(u)}{\sum_{i=0}^{n} w_i B_i^n(u)}, \quad u \in I \subset IR$$

The $w_i > 0,\ i = 0,\dots,n$ are called weights.

Homogeneous representation :

$$\mathbf{b}_H(u) = \sum_{i=0}^{n} \mathbf{b}_{H\,i}\ B_i^n(u), \quad u \in I \subset IR$$

with the homogeneous Bézier points

$$\mathbf{b}_{H\,i} = \begin{pmatrix} w_i \\ w_i \mathbf{b}_i \end{pmatrix}$$

Rational Bézier Curves: Properties and Algorithms

Properties:
- the same properties like polynomial curve, and
- projective invariance
- the weights are an additional design parameter

Algorithms
- All algorithms of polynomial Bézier curves can be applied without any change to the homogeneous representation of rational Bézier curves.

Non Uniform Rational B-Splines

A NURBS curve with respect to the control points $d_i,\ i = 0,\dots,n$ and the knot vector $U = (u_0,\dots,u_{n+k})$ is defined as

$$\mathbf{n}(u) = \frac{\sum_{i=0}^{n} w_i \mathbf{d}_i N_i^k(u)}{\sum_{i=0}^{n} w_i N_i^k(u)}, \quad u \in [u_0, u_{n+k}) \subset IR$$

The $w_i > 0,\ i = 0,\dots,n$ are called weights.

Homogeneous representation :

$$\mathbf{n}_H(u) = \sum_{i=0}^{n} \mathbf{n}_{H\,i}\ N_i^k(u), \quad u \in [u_0, u_{n+k}) \subset IR$$

with the homogeneous Bézier points

$$\mathbf{n}_{H\,i} = \begin{pmatrix} w_i \\ w_i \mathbf{n}_i \end{pmatrix}$$

NURBS: Properties and Algorithms

Properties:
- the same properties like polynomial curve, and
- projective invariance
- changing the weight w_i affects only the interval $[u_i, u_{i+k})$

Algorithms
- All algorithms of B-spline curves can be applied without any change to the homogeneous representation of NURBS curves.

Tensorproduct surfaces

"A surface is the locus of a curve that is moving through space and thereby changing the shape"

Given a curve

$$\mathbf{f}(u) = \sum_{i=0}^{n} \mathbf{c}_i F_i(u), \quad u \in I \subset IR$$

moving the control points yields

$$\mathbf{c}_i(v) = \sum_{j=0}^{m} \mathbf{a}_{ij} G_j(v), \quad v \in J \subset IR$$

Combining both yields a tensor product surface

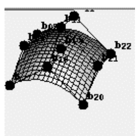
$$s(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{a}_{ij} F_i(u) G_j(v), \quad (u,v) \in I \times J \subset IR^2$$

Tensorproduct Bézier Surfaces

A tensorproduct Bézier surface is given by

$$\mathbf{b}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v), \quad (u,v) \in I \times J \subset IR^2$$

The Bézier points $\mathbf{b}_{ij}$ form the control net of the surface.



Tensor Product Bézier Surfaces Properties and Algorithms

Properties:

analogue to that of Bézier curves

Algorithms:

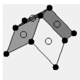
Apply algorithms for curves in two steps:

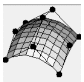
- Apply to

$$\mathbf{b}_i(v) = \sum_{j=0}^{m} \mathbf{b}_{ij} B_j^m(v), \quad i = 0,\ldots,n$$

- Apply to

$$\mathbf{b}(u,v) = \sum_{i=0}^{n} \mathbf{b}_i(v) B_i^n(u)$$





Tensorproduct B-Spline Surfaces

A tensorproduct B-spline surface with respect to the knot vectors

$U = (u_0,\ldots,u_{n+k}), V = (v_0,\ldots,v_{m+l})$

is given by

$$\mathbf{d}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{d}_{ij} N_i^k(u) N_j^l(v), \quad (u,v) \in [u_0, u_{n+k}] \times [v_0, v_{m+l}] \subset IR^2.$$

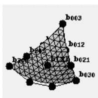
The control points $\mathbf{d}_{ij}$ form the control net of the surface.

Properties and Algorithms are analogue to the description for Bézier tensor product surfaces.

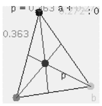
Bézier Triangles

A triangular Bézier patch is defined by

$$\mathbf{x}(u,v) = \sum_{\substack{i+j+k=1 \\ i,j,k>0}} \mathbf{b}_{ijk} B_{ijk}^n(u,v,w)$$

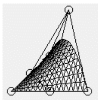


The u,v,w are baryzentric coordinates of the triangular parameter domain.



Generalized Bernstein polynomials:

$$B_{ijk}^n(u,v,w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k$$



Bézier Triangles: Properties and Algorithms

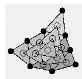
Properties:

- the same as in the univariate case

Algorithms:

- De Casteljau:

$$\mathbf{b}_{ijk}^l = u\mathbf{b}_{i+1\,jk}^{l-1} + v\mathbf{b}_{ij+1k}^{l-1} + w\mathbf{b}_{ijk+1}^{l-1}$$



- Subdivision
- Degree elevation